

PARABOLIC INDUCTIONS FOR PRO- p -IWAHORI HECKE ALGEBRAS

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ABSTRACT. We give some properties of parabolic inductions and their adjoint functors for pro- p -Iwahori Hecke algebras.

1. INTRODUCTION

Let F be a non-archimedean local field with residue characteristic p , G a connected reductive group over F and C a commutative ring. Fix a pro- p -Iwahori subgroup $I(1)$ of G and let \mathcal{H} be the space of $I(1)$ -biinvariant C -valued functions with compact support. Then via the convolution product, \mathcal{H} has the structure of an algebra, called *pro- p -Iwahori Hecke algebra*.

We are mainly interested in “mod p case”, namely when C is an algebraically closed field of characteristic p . In this case, this algebra has an important role to study modulo p representations of G . For example, in the proof of the classification theorem [AHHV], it has a crucial role. Motivated by this, we study the representation theory of \mathcal{H} .

As usual in the theory of reductive groups, the parabolic induction is one of the most important tool in the study of \mathcal{H} -modules. This functor is studied in [Oll10, Abe, Vig15b]. In particular, using parabolic induction, the simple \mathcal{H} -modules are classified in terms of supersingular modules [Abe, Theorem 1.1]. (Note that the supersingular modules are classified by Olivier [Oll14] and Vignéras [Vig15a].) This is an analogous of the main theorem of [AHHV].

In this paper, we study more details of the parabolic induction.

1.1. Results. Let P be a parabolic subgroup and \mathcal{H}_P the pro- p -Iwahori Hecke algebra of its Levi part. Then the parabolic induction I_P is defined by

$$I_P(\sigma) = \mathrm{Hom}_{(\mathcal{H}_P^-, j_P^{-*})}(\mathcal{H}, \sigma)$$

using “negative subalgebra” \mathcal{H}_P^- of \mathcal{H}_P and a certain algebra homomorphism $j_P^{-*}: \mathcal{H}_P^- \rightarrow \mathcal{H}$. Replacing “negative subalgebra” with “positive subalgebra” \mathcal{H}_P^+ , we also have another notion of “inductions”. Moreover, we also have other maps $j_P^\pm: \mathcal{H}_P^\pm \rightarrow \mathcal{H}$. Therefore, we have four “inductions”. We compare these inductions and give relations. It turns out that we only have two inductions I_P and $I'_P = \mathrm{Hom}_{(\mathcal{H}_P^-, j_P^-)}(\mathcal{H}, \cdot)$. The other inductions are described by I_P or I'_P (Proposition 4.13). As a corollary, we get a new description of I_P (Corollary 4.19).

The functor I_P has the left and right adjoint functors which are denoted by L_P and R_P , respectively. We calculate the image of simple modules of

\mathcal{H} by these functors. As proved in [Abe], any simple modules are got in the following way: starting with a simple supersingular module, take the generalized Steinberg module and apply the parabolic induction. So to calculate the image of simple modules of \mathcal{H} by these functors, it is sufficient to calculate the images of parabolic induction, generalized Steinberg modules and supersingular modules. We give such descriptions (Proposition 5.1, Corollary 5.8, Proposition 5.10, Proposition 5.11, Proposition 5.18). In particular, the image of simple modules by these functors is zero or again irreducible (Theorem 5.20). The same calculations for the representations of the group is given in [AHVa].

1.2. Applications. The main application is the calculation of the extension groups between simple \mathcal{H} -modules. Since the left adjoint functor L_P is exact, we have $\text{Ext}^i(\pi, I_P(\sigma)) \simeq \text{Ext}^i(L_P(\pi), \sigma)$ for \mathcal{H} -module π and \mathcal{H}_P -module σ . We also have a similar formula for the functor R_P . Since any simple \mathcal{H} -module is given by the parabolic induction from the generalized Steinberg modules, these formulas and results in this paper enable us to deduce the calculation of the extension groups to that of the extension groups between generalized Steinberg modules.

For the extension groups between the generalized Steinberg modules, it turns out that we need to study the functor I'_P . Especially, we need the relations between the functors I_P and I'_P . Such analysis and the calculation of extension groups will be appeared in sequel in which we use results in this paper.

1.3. Organization of the paper. In the next section we give notation and recall some results mainly from [Vig16]. In Section 3, we define a certain filtration on $I_P(\sigma)$. This filtration is analogous of the filtration of the parabolic induction of the group representation coming from the Bruhat cell. Some properties of the filtration are proved in [AHHV, AHVa] in the group case and we give analogous of them. These properties is used in Section 5. The study of four “inductions” is done in Section 4. In Section 5, we calculate the image of parabolic induction, generalized Steinberg modules and supersingular modules by L_P and R_P .

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2. PRELIMINARIES

2.1. Pro- p -Iwahori Hecke algebra. Let \mathcal{H} be a pro- p -Iwahori Hecke algebra over a commutative ring C [Vig16]. We study modules over \mathcal{H} in this paper. *In this paper, a module means a right module.* The algebra \mathcal{H} is defined with a combinatorial data $(W_{\text{aff}}, S_{\text{aff}}, \Omega, W, W(1), Z_\kappa)$ and a parameter (q, c) .

We recall the definitions. The data satisfy the following.

- $(W_{\text{aff}}, S_{\text{aff}})$ is a Coxeter system.
- Ω acts on $(W_{\text{aff}}, S_{\text{aff}})$.

- $W = W_{\text{aff}} \rtimes \Omega$.
- Z_κ is a finite commutative group.
- The group $W(1)$ is an extension of W by Z_κ , namely we have an exact sequence $1 \rightarrow Z_\kappa \rightarrow W(1) \rightarrow W \rightarrow 1$.

The subgroup Z_κ is normal in $W(1)$. Hence the conjugate action of $w \in W(1)$ induces an automorphism of Z_κ , hence of the group ring $C[Z_\kappa]$. We denote it by $c \mapsto w \cdot c$.

Let $\text{Ref}(W_{\text{aff}})$ be the set of reflections in W_{aff} and $\text{Ref}(W_{\text{aff}}(1))$ the inverse image of $\text{Ref}(W_{\text{aff}})$ in $W(1)$. The parameter (q, c) maps $q: S_{\text{aff}} \rightarrow C$ and $c: \text{Ref}(W_{\text{aff}}(1)) \rightarrow C[Z_\kappa]$ with the following conditions. (Here the image of s by q (resp. c) is denoted by q_s (resp. c_s).)

- For $w \in W$ and $s \in S_{\text{aff}}$, if $ws w^{-1} \in S_{\text{aff}}$ then $q_{ws w^{-1}} = q_s$.
- For $w \in W(1)$ and $s \in \text{Ref}(W_{\text{aff}}(1))$, $c_{ws w^{-1}} = w \cdot c_s$.
- For $s \in \text{Ref}(W_{\text{aff}}(1))$ and $t \in Z_\kappa$, we have $c_{ts} = t c_s$.

Let $S_{\text{aff}}(1)$ be the inverse image of S_{aff} in $W(1)$. For $s \in S_{\text{aff}}(1)$, we write q_s for $q_{\bar{s}}$ where $\bar{s} \in S_{\text{aff}}$ is the image of s . The length function on W_{aff} is denoted by ℓ and its inflation to W and $W(1)$ is also denoted by ℓ .

The C -algebra \mathcal{H} is a free C -module and has a basis $\{T_w\}_{w \in W(1)}$. The multiplication is given by

- (Quadratic relations) $T_s^2 = q_s T_{s^2} + c_s T_s$ for $s \in S_{\text{aff}}(1)$.
- (Braid relations) $T_{vw} = T_v T_w$ if $\ell(vw) = \ell(v) + \ell(w)$.

We extend $q: S_{\text{aff}} \rightarrow C$ to $q: W \rightarrow C$ as follows. For $w \in W$, take s_1, \dots, s_l and $u \in \Omega$ such that $w = s_1 \cdots s_l u$. Then put $q_w = q_{s_1} \cdots q_{s_l}$. From the definition, we have $q_{w^{-1}} = q_w$. We also put $q_w = q_{\bar{w}}$ for $w \in W(1)$ with the image \bar{w} in W .

2.2. The data from a group. Let F be a non-archimedean local field, κ its residue field, p its residue characteristic and G a connected reductive group over F . We can get the data in the previous subsection from G as follows. See [Vig16], especially 3.9 and 4.2 for the details.

Fix a maximal split torus S and denote the centralizer of S by Z . Let Z^0 be the unique parahoric subgroup of Z and $Z(1)$ its pro- p radical. Then the group $W(1)$ (resp. W) is defined by $W(1) = N_G(Z)/Z(1)$ ($W = N_G(Z)/Z^0$) where $N_G(Z)$ is the normalizer of Z in G . We also have $Z_\kappa = Z^0/Z(1)$. Let G' be the group generated by the unipotent radical of parabolic subgroups [AHHV, II.1] and W_{aff} the image of $G' \cap N_G(Z)$ in W . Then this is a Coxeter group. Fix a set of simple reflections S_{aff} . The group W has the natural length function and let Ω be the set of length zero elements in W . Then we get the data $(W_{\text{aff}}, S_{\text{aff}}, \Omega, W, W(1), Z_\kappa)$.

Consider the apartment attached to S and an alcove surrounded by the hyperplanes fixed by S_{aff} . Let $I(1)$ be the pro- p -Iwahori subgroup attached to this alcove. Then with $q_s = \#(I(1)\tilde{s}I(1)/I(1))$ for $s \in S_{\text{aff}}$ with a lift $\tilde{s} \in N_G(Z)$ and suitable c_s , the algebra \mathcal{H} is isomorphic to the Hecke algebra attached to $(G, I(1))$ [Vig16, Proposition 4.4].

In this paper, *the data* $(W_{\text{aff}}, S_{\text{aff}}, \Omega, W, W(1), Z_\kappa)$ *and the parameters* (q, c) *come from G in this way.* Let $W_{\text{aff}}(1)$ be the image of $G' \cap N_G(Z)$ in $W(1)$ and put $\mathcal{H}_{\text{aff}} = \bigoplus_{w \in W_{\text{aff}}(1)} CT_w$. This is a subalgebra of \mathcal{H} .

Proposition 2.1 ([Vig16, Proposition 4.4]). *Let $s \in \text{Ref}(W(1))$. Then we have $c_s = \sum_{t \in Z_\kappa} c_s(t) T_t$ for some $c_s(t) \in \mathbb{Z}$ such that $\sum_{t \in Z_\kappa} c_s(t) = q_s - 1$.*

2.3. Assumptions on C . In this paper, C is assumed to be a commutative ring. Sometimes we add the following assumptions.

- $p = 0$ in C .
- C is an algebraically closed field of characteristic p

We declare this assumption at the top of the subsection or the statement of a theorem/proposition/lemma etc. Otherwise we do not assume anything on C .

2.4. The algebra $\mathcal{H}[q_s]$ and $\mathcal{H}[q_s^{\pm 1}]$. For each $s \in S_{\text{aff}}$, let \mathbf{q}_s be an indeterminate such that if $ws w^{-1} \in S_{\text{aff}}$ for $w \in W$, we have $\mathbf{q}_{ws w^{-1}} = \mathbf{q}_s$. Let $C[\mathbf{q}_s]$ be a polynomial ring with these indeterminate. Then with the parameter $s \mapsto \mathbf{q}_s$ and the other data coming from G , we have the algebra. This algebra is denoted by $\mathcal{H}[\mathbf{q}_s]$ and we put $\mathcal{H}[\mathbf{q}_s^{\pm 1}] = \mathcal{H}[\mathbf{q}_s] \otimes_{C[\mathbf{q}_s]} C[\mathbf{q}_s^{\pm 1}]$. Under $\mathbf{q}_s \mapsto \#(I(1)\tilde{s}I(1)/I(1)) \in C$ where $\tilde{s} \in N_G(Z)$ is a lift of s , we have $\mathcal{H}[\mathbf{q}_s] \otimes_{C[\mathbf{q}_s]} C \simeq \mathcal{H}$. As an abbreviation, we denote \mathbf{q}_s by just q_s . Consequently we denote by $\mathcal{H}[q_s]$ (resp. $\mathcal{H}[q_s^{\pm 1}]$).

Since q_s is invertible in $\mathcal{H}[q_s^{\pm 1}]$, we can do some calculations in $\mathcal{H}[q_s^{\pm 1}]$ with q_s^{-1} . If the result can be stated in $\mathcal{H}[q_s]$, then this is an equality in $\mathcal{H}[q_s]$ since $\mathcal{H}[q_s]$ is a subalgebra of $\mathcal{H}[q_s^{\pm 1}]$ and by specializing, we can get some equality in \mathcal{H} . See [Vig16, 4.5] for more details.

2.5. The root system and the Weyl groups. Let $W_0 = N_G(Z)/Z$ be the finite Weyl group. Then this is a quotient of W . Recall that we have the alcove defining $I(1)$. Fix a special point \mathbf{x}_0 from the border of this alcove. Then $W_0 \simeq \text{Stab}_W \mathbf{x}_0$ and the inclusion $\text{Stab}_W \mathbf{x}_0 \hookrightarrow W$ is a splitting of the canonical projection $W \rightarrow W_0$. Throughout this paper, we fix this special point and regard W_0 as a subgroup of W . Set $S_0 = S_{\text{aff}} \cap W_0 \subset W$. This is a set of simple reflections in W_0 . For each $w \in W_0$, we fix a representative $n_w \in W(1)$ such that $n_{w_1 w_2} = n_{w_1} n_{w_2}$ if $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$.

The group W_0 is the Weyl group of the root system Σ attached to (G, S) . Our fixed alcove and special point give a positive system of Σ , denoted by Σ^+ . The set of simple roots is denoted by Δ . As usual, for $\alpha \in \Delta$, let $s_\alpha \in S_0$ be a simple reflection for α .

The kernel of $W(1) \rightarrow W_0$ (resp. $W \rightarrow W_0$) is denoted by $\Lambda(1)$ (resp. Λ). Then $Z_\kappa \subset \Lambda(1)$ and we have $\Lambda = \Lambda(1)/Z_\kappa$. The group Λ (resp. $\Lambda(1)$) is isomorphic to Z/Z^0 (resp. $Z/Z(1)$). Any element in $W(1)$ can be uniquely written as $n_w \lambda$ where $w \in W_0$ and $\lambda \in \Lambda(1)$. We have $W = W_0 \ltimes \Lambda$.

2.6. The map ν . The group W acts on the apartment attached to S and the action of Λ is by the translation. Since the group of translations of the apartment is $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$, we have a group homomorphism $\nu: \Lambda \rightarrow X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$. The compositions $\Lambda(1) \rightarrow \Lambda \rightarrow X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ and $Z \rightarrow \Lambda \rightarrow X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ are also denoted by ν . The homomorphism $\nu: Z \rightarrow X_*(S) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \text{Hom}_{\mathbb{Z}}(X^*(S), \mathbb{R})$ is characterized by the following: For $t \in S$ and $\chi \in X^*(S)$, we have $\nu(t)(\chi) = -\text{val}(\chi(t))$ where val is the normalized valuation of F . The kernel of $\nu: Z \rightarrow X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ is equal to the maximal

compact subgroup \tilde{Z} of Z . In particular, $\text{Ker}(\Lambda(1) \xrightarrow{\nu} X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}) = \tilde{Z}/Z(1)$ is a finite group.

We call $\lambda \in \Lambda(1)$ dominant (resp. anti-dominant) if $\nu(\lambda)$ is dominant (resp. anti-dominant).

Since the group W_{aff} is a Coxeter system, it has the Bruhat order denoted by \leq . For $w_1, w_2 \in W_{\text{aff}}$, we write $w_1 < w_2$ if there exists $u \in \Omega$ such that $w_1 u, w_2 u \in W_{\text{aff}}$ and $w_1 u < w_2 u$. Moreover, for $w_1, w_2 \in W(1)$, we write $w_1 < w_2$ if $w_1 \in W_{\text{aff}}(1)w_2$ and $\overline{w}_1 < \overline{w}_2$ where $\overline{w}_1, \overline{w}_2$ are the image of w_1, w_2 in W , respectively. We write $w_1 \leq w_2$ if $w_1 < w_2$ or $w_1 = w_2$.

2.7. Other basis. For $w \in W(1)$, take $s_1, \dots, s_l \in S_{\text{aff}}(1)$ and $u \in W(1)$ such that $l = \ell(w)$, $\ell(u) = 0$ and $w = s_1 \cdots s_l u$. Set $T_w^* = (T_{s_1} - c_{s_1}) \cdots (T_{s_l} - c_{s_l}) T_u$. Then this does not depend on the choice and we have $T_w^* \in T_w + \sum_{v < w} C T_v$. In particular, $\{T_w^*\}_{w \in W(1)}$ is a basis of \mathcal{H} . We also have the following other definition of T_w^* . In $\mathcal{H}[q_s^{\pm 1}]$, T_w is invertible and $q_w T_w^{-1} \in \mathcal{H}[q_s]$. The element T_w^* is the image of $q_w T_w^{-1}$ in \mathcal{H} .

For simplicity, we always assume that our commutative ring C contains a square root of q_s which is denoted by $q_s^{1/2}$ for $s \in S_{\text{aff}}$. For $w = s_1 \cdots s_l u$ where $\ell(w) = l$ and $\ell(u) = 0$, $q_w^{1/2} = q_{s_1}^{1/2} \cdots q_{s_l}^{1/2}$ is a square root of q_w . For a spherical orientation o , there is a basis $\{E_o(w)\}_{w \in W(1)}$ of \mathcal{H} introduced in [Vig16, 5]. We have

$$E_o(w) \in T_w + \sum_{v < w} C T_v.$$

This satisfies the following product formula [Vig16, Theorem 5.25].

$$(2.1) \quad E_o(w_1) E_{o \cdot w_1}(w_2) = q_{w_1 w_2}^{-1/2} q_{w_1}^{1/2} q_{w_2}^{1/2} E_o(w_1 w_2).$$

Remark 2.2. Since we do not assume that q_s is invertible in C , $q_{w_1 w_2}^{-1/2} q_{w_1}^{1/2} q_{w_2}^{1/2}$ does not make sense in a usual way. The meaning of this term is the following. Consider the indeterminate \mathbf{q}_s as in subsection 2.4. We also add $\mathbf{q}_s^{\pm 1/2}$. Then we have $\mathbf{q}_{w_1 w_2}^{-1/2} \mathbf{q}_{w_1}^{1/2} \mathbf{q}_{w_2}^{1/2}$. Then this belongs to $C[\mathbf{q}_s]$ [Vig16, Lemma 4.19]. Hence by specializing $\mathbf{q}_s \mapsto q_s$, we get $q_{w_1 w_2}^{-1/2} q_{w_1}^{1/2} q_{w_2}^{1/2} \in C$. Such calculations will be used several times in this paper implicitly. Note that since $\mathbf{q}_{w_1 w_2}^{-1/2} \mathbf{q}_{w_1}^{1/2} \mathbf{q}_{w_2}^{1/2}$ is in $C[\mathbf{q}_s]$, rather than $C[\mathbf{q}_s^{1/2}]$, the factor $q_{w_1 w_2}^{-1/2} q_{w_1}^{1/2} q_{w_2}^{1/2}$ does not depend on a choice of the square roots.

Applying (2.1) to $w_1 = w^{-1}$ and $w_2 = w$, if q_w is invertible then $E_{o \cdot w}(w)$ is invertible and we have

$$(2.2) \quad E_o(w^{-1}) = q_w E_{o \cdot w}(w)^{-1}.$$

There is a one-to-one correspondence between spherical orientations and closed Weyl chambers. Let o be a spherical orientation and \mathcal{D} the corresponding closed Weyl chamber. Then for $\lambda \in \Lambda(1)$, we have [Vig16, Example 5.30]

$$(2.3) \quad E_o(\lambda) = \begin{cases} T_\lambda & (\nu(\lambda) \in \mathcal{D}), \\ T_\lambda^* & (-\nu(\lambda) \in \mathcal{D}). \end{cases}$$

Let o_+ (resp. o_-) be the dominant (resp. anti-dominant) orientation. The orientation o_+ (resp. o_-) corresponds to the dominant (resp. anti-dominant)

chamber. Then for $s \in S_0$ and $v \in W_0$ [Vig16, Example 5.33],

$$(2.4) \quad E_{o_+ \cdot v}(n_s) = \begin{cases} T_{n_s} & (vs < v), \\ T_{n_s}^* & (vs > v), \end{cases} \quad E_{o_- \cdot v}(n_s) = \begin{cases} T_{n_s} & (vs > v), \\ T_{n_s}^* & (vs < v). \end{cases}$$

By the definition of spherical orientations, $o \cdot \lambda = o$ for any spherical orientation o and $\lambda \in \Lambda(1)$. Hence by (2.1), the subspace $\bigoplus_{\lambda \in \Lambda(1)} CE_o(\lambda)$ is a subalgebra of \mathcal{H} . We denote this subalgebra by \mathcal{A}_o .

Finally we introduce one more basis defined by

$$E_-(n_w \lambda) = q_{n_w \lambda}^{1/2} q_{n_w}^{-1/2} q_\lambda^{-1/2} T_{n_w}^* E_{o_-}(\lambda)$$

for $w \in W_0$ and $\lambda \in \Lambda(1)$. By [Abe, Lemma 4.2], $\{E_-(w) \mid w \in W(1)\}$ is a C -basis of \mathcal{H} .

2.8. Levi subalgebra. Since we have a positive system Σ^+ , we have a minimal parabolic subgroup B with a Levi part Z . In this paper, *parabolic subgroups are always standard, namely containing B* . Note that such parabolic subgroups correspond to subsets of Δ .

Let P be a parabolic subgroup. Attached to the Levi part of P containing Z , we have a data $(W_{\text{aff},P}, S_{\text{aff},P}, \Omega_P, W_P, W_P(1), Z_\kappa)$ and parameters (q_P, c_P) . Hence we have the algebra \mathcal{H} . The parameter c_P is given by the restriction of c , hence we denote it just by c . The parameter q_P is defined as in [Abe, 4.1].

For the objects attached to this data, we add a suffix P . We have the set of simple roots Δ_P , the root system Σ_P and its positive system Σ_P^+ , the finite Weyl group $W_{0,P}$, the set of simple reflections $S_{0,P} \subset W_{0,P}$, the length function ℓ_P and the base $\{T_w^P\}_{w \in W_P(1)}$, $\{T_w^{P*}\}_{w \in W_P(1)}$ and $\{E_o^P(w)\}_{w \in W_P(1)}$ of \mathcal{H}_P . Note that we have no Λ_P , $\Lambda_P(1)$ and $Z_{\kappa,P}$ since they are equal to Λ , $\Lambda(1)$ and Z_κ . We have the following lemma. This follows from [Vig16, 3.8, 4.2]. See also [AHVb, Remark 4.6].

Lemma 2.3. *Let $s \in S_{\text{aff}}$, $\tilde{s} \in S_{\text{aff}}(1)$ its lift, $\alpha \in \Sigma$ such that the image of s in W_0 is s_α and P a parabolic subgroup such that $\alpha \in \Sigma_P$.*

- (1) *We can take \tilde{s} from $W_{\text{aff},P}(1)$.*
- (2) *If $\tilde{s} \in W_{\text{aff},P}(1)$, then $c_{\tilde{s}} \in C[Z_\kappa \cap W_{\text{aff},P}(1)]$.*

In particular, for $s = s_\alpha \in S_0$ where $\alpha \in \Delta$, we can take (and do) $n_s \in W_{\text{aff},P_\alpha}(1)$ where P_α is a parabolic subgroup such that $\Delta_{P_\alpha} = \{\alpha\}$.

An element $w = n_v \lambda \in W_P(1)$ where $v \in W_{0,P}$ and $\lambda \in \Lambda(1)$ is called P -positive (resp. P -negative) if $\langle \alpha, \nu(\lambda) \rangle \leq 0$ (resp. $\langle \alpha, \nu(\lambda) \rangle \geq 0$) for any $\alpha \in \Sigma^+ \setminus \Sigma_P^+$. Let $W_P^+(1)$ (resp. $W_P^-(1)$) be the set of P -positive (resp. P -negative) elements and put $\mathcal{H}_P^\pm = \bigoplus_{w \in W_P^\pm(1)} CT_w^P$. These are subalgebras of \mathcal{H}_P [Abe, Lemma 4.1]. In general, for a group Γ , we denote its center by $Z(\Gamma)$.

Lemma 2.4. *There exists λ_P^+ (resp. λ_P^-) in the center of $W_P(1)$ such that $\langle \alpha, \nu(\lambda_P^+) \rangle < 0$ (resp. $\langle \alpha, \nu(\lambda_P^-) \rangle > 0$) for all $\alpha \in \Sigma^+ \setminus \Sigma_P^+$.*

Proof. We prove the existence of λ_P^+ . Take $\lambda_0 \in Z(\Lambda(1))$ such that the stabilizers of $\nu(\lambda_0)$ in W_0 is $W_{0,P}$ and $\langle \alpha, \nu(\lambda_0) \rangle < 0$ for any $\alpha \in \Sigma^+ \setminus \Sigma_P^+$. Fix $w \in W_{0,P}$. Then for each $k \in \mathbb{Z}_{\geq 0}$, $\nu(n_w \lambda_0^k n_w^{-1} \lambda_0^{-k}) = 0$. Namely

$n_w \lambda_0^k n_w^{-1} \lambda_0^{-k} \in \text{Ker } \nu$. Since $\text{Ker } \nu$ is finite, there exists $k_1 > k_2$ such that $n_w \lambda_0^{k_1} n_w^{-1} \lambda_0^{-k_1} = n_w \lambda_0^{k_2} n_w^{-1} \lambda_0^{-k_2}$. Hence $n_w \lambda_0^{k_1-k_2} n_w^{-1} \lambda_0^{-(k_1-k_2)} = 1$. Therefore, for each $w \in W_{0,P}$, there exists $k_w \in \mathbb{Z}_{>0}$ such that $n_w \lambda_0^{k_w} n_w^{-1} \lambda_0^{-k_w} = 1$. Hence $n_w \cdot \lambda_0^{k_w} = \lambda_0^{k_w}$. Set $k = \prod_{w \in W_{0,P}} k_w$ and $\lambda = \lambda_0^k$. Then for $w \in W_{0,P}$, we have $n_w \cdot \lambda = \lambda$. Since we took λ_0 from the center of $\Lambda(1)$, λ commutes with the elements in $\Lambda(1)$. The group $W_P(1)$ is generated by $\{n_w \mid w \in W_{0,P}\}$ and $\Lambda(1)$. Hence $\lambda \in Z(W_P(1))$. For $\alpha \in \Sigma^+ \setminus \Sigma_P^+$, we have $\langle \alpha, \nu(\lambda) \rangle = k \langle \alpha, \nu(\lambda_0) \rangle < 0$. Therefore $\lambda_P^+ = \lambda$ satisfies the condition of the lemma. The element $\lambda_P^- = (\lambda_P^+)^{-1}$ satisfies the condition of the lemma. \square

Proposition 2.5 ([Vig15b, Theorem 1.4]). *Let λ_P^+ (resp. λ_P^-) be in the center of $W_P(1)$ such that $\langle \alpha, \nu(\lambda_P^+) \rangle < 0$ (resp. $\langle \alpha, \nu(\lambda_P^-) \rangle > 0$) for all $\alpha \in \Sigma^+ \setminus \Sigma_P^+$. Then $T_{\lambda_P^+}^P = T_{\lambda_P^+}^{P*} = E_{o_{-,P}}^P(\lambda_P^+)$ (resp. $T_{\lambda_P^-}^P = T_{\lambda_P^-}^{P*} = E_{o_{-,P}}^P(\lambda_P^-)$) is in the center of \mathcal{H}_P and we have $\mathcal{H}_P = \mathcal{H}_P^+ E_{o_{-,P}}^P(\lambda_P^+)^{-1}$ (resp. $\mathcal{H}_P = \mathcal{H}_P^- E_{o_{-,P}}^P(\lambda_P^-)^{-1}$).*

We define $j_P^\pm: \mathcal{H}_P^\pm \rightarrow \mathcal{H}$ and $j_P^{\pm*}: \mathcal{H}_P^\pm \rightarrow \mathcal{H}$ by $j_P^\pm(T_w^P) = T_w$ and $j_P^{\pm*}(T_w^{P*}) = T_w^*$ for $w \in W_P^\pm(1)$. Then these are algebra homomorphisms. For j_P^+ and j_P^{+*} , it is [Abe, Lemma 4.6] and the same argument can apply for j_P^{+*} and j_P^- .

Some special cases of the following lemma is proved in [Oll15, Abe].

Lemma 2.6. *Let $w \in W_P(1)$.*

- (1) *If the image of w in W_P is in $W_{0,P}$, then w is both P -positive and P -negative. Moreover we have $j_P^\pm(T_w^{P*}) = T_w^*$ and $j_P^{\pm*}(T_w^P) = T_w$.*
- (2) *For $x \in W_{0,P}$, we have*

$$\begin{aligned} j_P^+(E_{o_{-,P} \cdot x}^P(w)) &= E_{o_{-} \cdot x}(w) \quad (w \in W_P^+(1)), \\ j_P^{+*}(E_{o_{+,P} \cdot x}^P(w)) &= E_{o_{+} \cdot x}(w) \quad (w \in W_P^+(1)), \\ j_P^-(E_{o_{+,P} \cdot x}^P(w)) &= E_{o_{+} \cdot x}(w) \quad (w \in W_P^-(1)), \\ j_P^{-*}(E_{o_{-,P} \cdot x}^P(w)) &= E_{o_{-} \cdot x}(w) \quad (w \in W_P^-(1)). \end{aligned}$$

Proof. (1) Let v be the image of w in W_P . Then it is in $W_{0,P}$ by the assumption. Set $t = n_v^{-1}w$. Then $t \in \text{Ker}(W_P(1) \rightarrow W_P) = Z_\kappa$. In particular $t \in \Lambda(1)$ and we have $w = n_v t$. Since $\nu(t) = 0$, w is P -positive and P -negative.

For the proof of the second statement, first assume that $w = t \in Z_\kappa$. Since $\ell(t) = 0$, we have $T_t^P = T_t^{P*}$ and $T_t = T_t^*$. Therefore we get the statement. In particular, for $s \in S_{\text{aff}}(1)$, we have $j_P^\pm(c_s) = j_P^{\pm*}(c_s) = c_s$.

To prove the second statement, by induction on $\ell_P(w)$, we may assume that the image of w in $W_{0,P}$ is a simple reflection. Set $s = w$. Then $j_P^\pm(T_s^{P*}) = j_P^\pm(T_s^P - c_s) = T_s - c_s$. Since the image of s in W_P is a finite simple reflection, the image of s in W is also a finite simple reflection. Therefore $T_s^* = T_s - c_s$. Hence $j_P^\pm(T_s^{P*}) = T_s^*$. Similarly we have $j_P^{\pm*}(T_s^P) = j_P^{\pm*}(T_s^{P*} + c_s) = T_s^* + c_s = T_s$.

(2) It is sufficient to prove the lemma in $\mathcal{H}[q_s^{\pm 1}]$. First we prove the following: Let w_1, w_2, w_3 be P -negative or P -positive elements such that

$w_3 = w_1 w_2$. Then if the lemma holds for $w = w_1, w_2$ (resp. $w = w_2, w_3$) then it also holds for $w = w_3$ (resp. $w = w_1$). We prove this claim only for j_P^+ . For the other cases, the same proof apply.

For j_P^+ , we assume that w_1, w_2, w_3 are P -positive. By (2.1), we have

$$E_{o_-, P \cdot x}^P(w_1) E_{o_-, P \cdot n_x w_1}^P(w_2) = q_{P, w_1}^{1/2} q_{P, w_2}^{1/2} q_{P, w_3}^{-1/2} E_{o_-, P \cdot x}^P(w_3).$$

Since j_P^+ is an algebra homomorphism, we have

$$j_P^+(E_{o_-, P \cdot x}^P(w_1)) j_P^+(E_{o_-, P \cdot n_x w_1}^P(w_2)) = q_{P, w_1}^{1/2} q_{P, w_2}^{1/2} q_{P, w_3}^{-1/2} j_P^+(E_{o_-, P \cdot x}^P(w_3)).$$

By [Abe, Lemma 4.5], we have $q_{P, w_1}^{1/2} q_{P, w_2}^{1/2} q_{P, w_3}^{-1/2} = q_{w_1}^{1/2} q_{w_2}^{1/2} q_{w_3}^{-1/2}$. Hence

$$j_P^+(E_{o_-, P \cdot x}^P(w_1)) j_P^+(E_{o_-, P \cdot n_x w_1}^P(w_2)) = q_{w_1}^{1/2} q_{w_2}^{1/2} q_{w_3}^{-1/2} j_P^+(E_{o_-, P \cdot x}^P(w_3)).$$

If the lemma holds for $w = w_1$ and w_2 , then

$$E_{o_-, x}(w_1) E_{o_-, n_x w_1}(w_2) = q_{w_1}^{1/2} q_{w_2}^{1/2} q_{w_3}^{-1/2} j_P^+(E_{o_-, P \cdot x}^P(w_3)).$$

By (2.1), we get $j_P^+(E_{o_-, P \cdot x}^P(w_3)) = E_{o_-, x}(w_3)$. If the lemma holds for $w = w_2$ and w_3 , then

$$j_P^+(E_{o_-, P \cdot x}^P(w_1)) E_{o_-, n_x w_1}(w_2) = q_{w_1}^{1/2} q_{w_2}^{1/2} q_{w_3}^{-1/2} E_{o_-, x}(w_3).$$

By (2.2), $E_{o_-, P \cdot n_x w_1}(w_2)$ is invertible in $\mathcal{H}[q_s^{\pm 1}]$. Hence by (2.1), we get $j_P^+(E_{o_-, P \cdot x}^P(w_1)) = E_{o_-, x}(w_1)$. The same proof can be applicable for the other cases.

We assume that $w = n_v$ for $v \in W_{0, P}$. By the above argument, we may assume that $v \in S_{0, P}$. We write s for v . By (2.4), we have

$$E_{o_+, P \cdot x}^P(n_s) = \begin{cases} T_{n_s}^P & (xs < x), \\ T_{n_s}^{P*} & (xs > x), \end{cases} \quad E_{o_-, P \cdot x}^P(n_s) = \begin{cases} T_{n_s}^P & (xs > x), \\ T_{n_s}^{P*} & (xs < x). \end{cases}$$

Hence by (1), for $j = j_P^{+*}$ or j_P^- and $j' = j_P^+$ or j_P^{-*} , we have

$$j(E_{o_+, P \cdot x}^P(n_s)) = \begin{cases} T_{n_s} & (xs < x), \\ T_{n_s}^* & (xs > x), \end{cases} \quad j'(E_{o_-, P \cdot x}^P(n_s)) = \begin{cases} T_{n_s} & (xs > x), \\ T_{n_s}^* & (xs < x). \end{cases}$$

On the other hand, we have

$$E_{o_+, x}(n_s) = \begin{cases} T_{n_s} & (xs < x), \\ T_{n_s}^* & (xs > x), \end{cases} \quad E_{o_-, x}(n_s) = \begin{cases} T_{n_s} & (xs > x), \\ T_{n_s}^* & (xs < x). \end{cases}$$

We get the lemma in this case.

Next we assume that $w = \lambda \in \Lambda(1)$ and it is in a chamber corresponding to the spherical orientation. We deal with 4 cases separately. Note that since $x(\Sigma^+ \setminus \Sigma_P^+) = \Sigma^+ \setminus \Sigma_P^+$, $n_x \cdot \lambda$ is P -positive (resp. P -negative) if and only if λ is P -positive (reps. P -negative).

j_P^+ : Assume that $n_x \cdot \lambda$ is anti-dominant. Then $n_x \cdot \lambda$ is P -positive. Hence λ is also P -positive. We have $E_{o_-, P \cdot x}^P(\lambda) = T_\lambda^P$ and $E_{o_-, x}(\lambda) = T_\lambda$ by (2.3). Hence $j_P^+(E_{o_-, P \cdot x}^P(\lambda)) = E_{o_-, x}(\lambda)$ in this case.

j_P^{+*} : Assume that $n_x \cdot \lambda$ is anti-dominant. Then $n_x \cdot \lambda$ is P -positive. Hence λ is also P -positive. We have $E_{o_+, P \cdot x}^P(\lambda) = T_\lambda^{P*}$ and $E_{o_+, x}(\lambda) = T_\lambda^*$ by (2.3). Hence $j_P^{+*}(E_{o_+, P \cdot x}^P(\lambda)) = E_{o_+, x}(\lambda)$ in this case.

j_P^- : Assume that $n_x \cdot \lambda$ is dominant. Then $n_x \cdot \lambda$ is P -negative. Hence λ is also P -negative. We have $E_{o_+,P}^P(\lambda) = T_\lambda^P$ and $E_{o_+,x}(\lambda) = T_\lambda$ by (2.3). Hence $j_P^-(E_{o_+,P}^P(\lambda)) = E_{o_+,x}(\lambda)$ in this case.

j_P^{-*} : Assume that $n_x \cdot \lambda$ is dominant. Then $n_x \cdot \lambda$ is P -negative. Hence λ is also P -negative. We have $E_{o_-,P}^P(\lambda) = T_\lambda^{P*}$ and $E_{o_-,x}(\lambda) = T_\lambda^*$ by (2.3). Hence $j_P^{-*}(E_{o_-,P}^P(\lambda)) = E_{o_-,x}(\lambda)$ in this case.

We prove the general case for j_P^+ . The same argument implies the other cases. Let $w \in W_P^+(1)$ and take $v \in W_0$, $\lambda_1, \lambda_2 \in \Lambda(1) \cap W_P^+(1)$ such that $w = n_v \lambda_1 \lambda_2^{-1}$, $n_x \cdot \lambda_1$ and $n_x \cdot \lambda_2$ are anti-dominant. Then the lemma holds for n_v, λ_1 and λ_2 . Hence the argument in the beginning of the proof of (2), we get the lemma for w . \square

Corollary 2.7. *If $w \in W_P(1)$ is both P -positive and P -negative (in particular, if $w = n_v$ for some $v \in W_{0,P}$), then we have $j_P^\pm(T_w^{P*}) = T_w^*$ and $j_P^{\pm*}(T_w^P) = T_w$.*

Proof. We only prove $j_P^+(T_w^{P*}) = T_w^*$. The same argument apply to other cases.

Take $c_v \in C$ such that $T_w^{P*} = \sum_v c_v E_{o_-,P}^P(v)$. If $c_v \neq 0$ then $v \leq w$ in $W_P(1)$. Hence v is also P -positive and P -negative by [Abe, Lemma 4.1]. We have

$$\begin{aligned} j_P^+(T_w^{P*}) &= \sum_v c_v j_P^+(E_{o_-,P}^P(v)) \\ &= \sum_v c_v E_{o_-}(v) \\ &= \sum_v c_v j_P^{-*}(E_{o_-,P}^P(v)) \\ &= j_P^{-*}(T_w^{P*}) = T_w^*. \end{aligned}$$

We get the corollary. \square

We also use the following relative setting. Let Q be a parabolic subgroup containing P and let $W_P^{Q+}(1)$ (resp. $W_P^{Q-}(1)$) be the set of $n_w \lambda$ where $\langle \alpha, \nu(\lambda) \rangle \leq 0$ (resp. $\langle \alpha, \nu(\lambda) \rangle \geq 0$) for any $\alpha \in \Sigma_Q^+ \setminus \Sigma_P^+$ and $w \in W_{0,P}$. Put $\mathcal{H}_P^{Q\pm} = \bigoplus_{w \in W_P^{Q\pm}(1)} CT_w^P \subset \mathcal{H}_P$. Then we have homomorphisms $j_P^{Q\pm}, j_P^{Q\pm*}: \mathcal{H}_P^{Q\pm} \rightarrow \mathcal{H}_Q$ defined by a similar way.

2.9. Parabolic induction. Let P be a parabolic subgroup and σ an \mathcal{H}_P -module. (This is a right module as in subsection 2.1.) Then we define an \mathcal{H} -module $I_P(\sigma)$ by

$$I_P(\sigma) = \text{Hom}_{(\mathcal{H}_P, j_P^{-*})}(\mathcal{H}, \sigma).$$

We call I_P the parabolic induction. For $P \subset P_1$, we write

$$I_P^{P_1}(\sigma) = \text{Hom}_{(\mathcal{H}_P^{P_1-}, j_P^{P_1-*})}(\mathcal{H}_{P_1}, \sigma).$$

Remark 2.8. Since we have two algebras \mathcal{H}_P^\pm and four homomorphisms $j_P^\pm, j_P^{\pm*}$, we can define four ‘‘inductions’’. The other inductions are studied in Section 4.

We recall some properties from [Abe] and [Vig15b]. Let P be a parabolic subgroup. Set $W_0^P = \{w \in W_0 \mid w(\Delta_P) \subset \Sigma^+\}$. Then the multiplication map $W_0^P \times W_{0,P} \rightarrow W_0$ is bijective and for $w_1 \in W_0^P$ and $w_2 \in W_{0,P}$, we have $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. We also put ${}^P W_0 = \{w \in W_0 \mid w^{-1}(\Delta_P) \subset \Sigma^+\}$. Then the multiplication map $W_{0,P} \times {}^P W_0 \rightarrow W_0$ is bijective and for $w_1 \in W_{0,P}$ and $w_2 \in {}^P W_0$, we have $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$.

Proposition 2.9. *Let P be a parabolic subgroup and σ an \mathcal{H}_P -module.*

- (1) *The map $I_P(\sigma) \ni \varphi \mapsto (\varphi(T_{n_w}))_{w \in W_0^P} \in \bigoplus_{w \in W_0^P} \sigma$ is bijective.*
- (2) *The map $I_P(\sigma) \ni \varphi \mapsto (\varphi(T_{n_w}^*))_{w \in W_0^P} \in \bigoplus_{w \in W_0^P} \sigma$ is bijective.*

Proof. The first part is [Vig15b, Lemma 3.9]. The second part follows from the first part and $T_{n_w}^* \in T_{n_w} + \sum_{v < w} T_{n_v} C[Z_\kappa]$ with a usual triangular argument. \square

Let $P \subset Q$ be parabolic subgroups. We have an \mathcal{H}_Q -module $I_P^Q(\sigma)$ for an \mathcal{H}_P -module σ . The parabolic inductions are transitive.

Proposition 2.10 ([Vig15b, Proposition 4.10]). *The map $\varphi \mapsto (X \mapsto \varphi(X)(1))$ gives an isomorphism $I_Q(I_P^Q(\sigma)) \simeq I_P(\sigma)$.*

2.10. Length. We prove some lemmas on the length. These lemmas follow from the formula on the length [Vig16].

We recall the formula. Let $\Sigma' \subset \text{Hom}_{\mathbb{R}}(X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}, \mathbb{R})$ be the subset such that the action of $\text{Ref}(W_{\text{aff}})$ on $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ is the set of reflections with respect to the hyperplanes $\{\{v \in X_*(S) \otimes_{\mathbb{Z}} \mathbb{R} \mid \alpha(v) + k = 0\} \mid \alpha \in \Sigma', k \in \mathbb{Z}\}$. Then for any $\alpha \in \Sigma'$, there exists only one element in $\mathbb{R}_{>0} \alpha \cap \Sigma_{\text{red}}$ and the map α sends to this unique element gives a bijection $\Sigma' \simeq \Sigma_{\text{red}}$ where Σ_{red} is the set of reduced roots in Σ . Set $\Sigma'^+ = \{\alpha \in \Sigma' \mid \Sigma^+ \cap \mathbb{R}_{>0} \alpha \neq \emptyset\}$. Then Σ' is a root system with the Weyl group W_0 and Σ'^+ is a positive system of Σ' . For $v \in W_0$ and $\lambda \in \Lambda(1)$, we have [Vig16, Corollary 5.10]

$$\begin{aligned} \ell(\lambda n_v) &= \sum_{\alpha \in \Sigma'^+ \cap v(\Sigma'^+)} |\langle \alpha, \nu(\lambda) \rangle| + \sum_{\alpha \in \Sigma'^+ \cap v(\Sigma'^-)} |\langle \alpha, \nu(\lambda) \rangle - 1|, \\ \ell(n_v \lambda) &= \sum_{\alpha \in \Sigma'^+ \cap v^{-1}(\Sigma'^+)} |\langle \alpha, \nu(\lambda) \rangle| + \sum_{\alpha \in \Sigma'^+ \cap v^{-1}(\Sigma'^-)} |\langle \alpha, \nu(\lambda) \rangle + 1|. \end{aligned}$$

The map $W \rightarrow \text{Aut}(X_*(S) \otimes_{\mathbb{Z}} \mathbb{R})$ is injective on W_{aff} and the image of W_{aff} is the Weyl group of the affine root system $\{\alpha + k \mid \alpha \in \Sigma', k \in \mathbb{Z}\}$. Hence we have $W_{\text{aff}} \simeq W_0 \ltimes \mathbb{Z}\Sigma'^{\vee}$. For $\lambda \in \Lambda(1) \cap W_{\text{aff}}(1)$, $\nu(\lambda)$ is in the image of $\Lambda \cap W_{\text{aff}}$. It is $\mathbb{Z}\Sigma'^{\vee}$. Hence $\nu(\lambda) \in \mathbb{Z}\Sigma'^{\vee} \subset \mathbb{R}\Sigma^{\vee}$.

Lemma 2.11. *Let $\lambda_1, \lambda_2 \in \Lambda(1)$. We have $\ell(\lambda_1 \lambda_2) = \ell(\lambda_1) + \ell(\lambda_2)$ if and only if the vectors $\nu(\lambda_1), \nu(\lambda_2)$ are in the same closed Weyl chamber, namely $\langle \alpha, \nu(\lambda_1) \rangle \langle \alpha, \nu(\lambda_2) \rangle \geq 0$ for any $\alpha \in \Sigma^+$.*

Proof. Obvious from the above formula. \square

Lemma 2.12. *Let $\lambda \in \Lambda(1)$. Then we have $\ell(\lambda) = 0$ if and only if $\langle \alpha, \nu(\lambda) \rangle = 0$ for any $\alpha \in \Sigma$.*

Proof. Obvious from the above formula. \square

Lemma 2.13. *If $w \in W(1)$ is in the center of $W(1)$, then $w \in \Lambda(1)$ and $\ell(w) = 0$.*

Proof. If w is in the center, then $w \in \Lambda(1)$ by [Vig14, Lemma 1.1]. For $\alpha \in \Sigma$, we have $n_{s_\alpha} w n_{s_\alpha}^{-1} = w$. Applying ν , we get $s_\alpha(\nu(w)) = \nu(w)$. Hence $\langle \alpha, \nu(w) \rangle = 0$. \square

Lemma 2.14. *If $\lambda \in \Lambda(1) \cap W_{\text{aff}}(1)$, then $\ell(\lambda)$ is even.*

Proof. We have $\ell(\lambda) = \sum_{\alpha \in \Sigma'^+} |\langle \alpha, \nu(\lambda) \rangle| \equiv \sum_{\alpha \in \Sigma'^+} \langle \alpha, \nu(\lambda) \rangle = 2\langle \rho, \nu(\lambda) \rangle \pmod{2}$ where $\rho = (1/2) \sum_{\alpha \in \Sigma'^+} \alpha$. For each $\alpha \in \Sigma'$ which is simple, $\langle \rho, \alpha^\vee \rangle = 1$. Hence for any $v \in \mathbb{Z}\Sigma'^\vee$, we have $\langle \rho, v \rangle \in \mathbb{Z}$. Hence $2\langle \rho, \nu(\lambda) \rangle$ is even. \square

Lemma 2.15. *For $w \in W(1)$ and $\lambda \in \Lambda(1)$, we have $\ell(w \cdot \lambda) = \ell(\lambda)$.*

Proof. Let $x \in W_0$ be the image of w . Then we have $\nu(w \cdot \lambda) = x(\nu(\lambda))$. Hence $\ell(w \cdot \lambda) = \sum_{\alpha \in \Sigma'^+} |\langle \alpha, x(\nu(\lambda)) \rangle| = \sum_{\alpha \in x^{-1}(\Sigma'^+)} |\langle \alpha, \nu(\lambda) \rangle|$. We have $x^{-1}(\Sigma'^+) = (\Sigma'^+ \cap x^{-1}(\Sigma'^+)) \sqcup (\Sigma'^- \cap x^{-1}(\Sigma'^+)) = (\Sigma'^+ \cap x^{-1}(\Sigma'^+)) \sqcup (-(\Sigma'^+ \cap x^{-1}(\Sigma'^-)))$. Since

$$\sum_{\alpha \in -(\Sigma'^+ \cap x^{-1}(\Sigma'^-))} |\langle \alpha, \nu(\lambda) \rangle| = \sum_{\alpha \in (\Sigma'^+ \cap x^{-1}(\Sigma'^-))} |\langle \alpha, \nu(\lambda) \rangle|,$$

we have

$$\begin{aligned} \ell(w \cdot \lambda) &= \sum_{\alpha \in (\Sigma'^+ \cap x^{-1}(\Sigma'^+))} |\langle \alpha, \nu(\lambda) \rangle| + \sum_{\alpha \in -(\Sigma'^+ \cap x^{-1}(\Sigma'^-))} |\langle \alpha, \nu(\lambda) \rangle| \\ &= \sum_{\alpha \in (\Sigma'^+ \cap x^{-1}(\Sigma'^+))} |\langle \alpha, \nu(\lambda) \rangle| + \sum_{\alpha \in (\Sigma'^+ \cap x^{-1}(\Sigma'^-))} |\langle \alpha, \nu(\lambda) \rangle| \\ &= \sum_{\alpha \in \Sigma'^+} |\langle \alpha, \nu(\lambda) \rangle| = \ell(\lambda). \end{aligned}$$

We finish the proof. \square

Lemma 2.16. *Let $v \in W_0$ and $\lambda \in \Lambda(1)$. Then we have*

$$\begin{aligned} \ell(\lambda n_v) &= \ell(\lambda) + \ell(v) - 2\#\{\alpha \in \Sigma_{\text{red}}^+ \mid v^{-1}(\alpha) < 0, \langle \alpha, \nu(\lambda) \rangle > 0\} \\ &= \ell(\lambda) - \ell(v) + 2\#\{\alpha \in \Sigma_{\text{red}}^+ \mid v^{-1}(\alpha) < 0, \langle \alpha, \nu(\lambda) \rangle \leq 0\} \end{aligned}$$

and

$$\begin{aligned} \ell(n_v \lambda) &= \ell(\lambda) + \ell(v) - 2\#\{\alpha \in \Sigma_{\text{red}}^+ \mid v(\alpha) < 0, \langle \alpha, \nu(\lambda) \rangle < 0\} \\ &= \ell(\lambda) - \ell(v) + 2\#\{\alpha \in \Sigma_{\text{red}}^+ \mid v(\alpha) < 0, \langle \alpha, \nu(\lambda) \rangle \geq 0\}. \end{aligned}$$

Proof. The formula for $\ell(n_v \lambda)$ follows from the formula for $\ell(\lambda n_v)$ and $\ell(n_v \lambda) = \ell((n_v \lambda)^{-1}) = \ell(\lambda^{-1} n_v^{-1}) = \ell(\lambda^{-1} n_{v^{-1}})$, here, at the last point, we use the fact that $n_{v^{-1}} n_v \in \text{Ker}(W(1) \rightarrow W)$ has the length zero.

Since

$$|\langle \alpha, \nu(\lambda) \rangle| - 1 = \begin{cases} |\langle \alpha, \nu(\lambda) \rangle| - 2 & (\langle \alpha, \nu(\lambda) \rangle > 0), \\ |\langle \alpha, \nu(\lambda) \rangle| & (\langle \alpha, \nu(\lambda) \rangle \leq 0), \end{cases}$$

we have

$$\begin{aligned} \ell(\lambda n_v) - \#(\Sigma'^+ \cap v(\Sigma'^-)) \\ = \sum_{\alpha \in \Sigma'^+} |\langle \alpha, \nu(\lambda) \rangle| - 2\#\{\alpha \in \Sigma'^+ \cap v(\Sigma'^-) \mid \langle \alpha, \nu(\lambda) \rangle > 0\}. \end{aligned}$$

We have $\#(\Sigma'^+ \cap v(\Sigma'^-)) = \ell(v)$ and $\sum_{\alpha \in \Sigma'^+} |\langle \alpha, \nu(\lambda) \rangle| = \ell(\lambda)$. Hence $\ell(\lambda n_v) = \ell(\lambda) + \ell(v) - 2\#\{\alpha \in \Sigma'^+ \cap v(\Sigma'^-) \mid \langle \alpha, \nu(\lambda) \rangle > 0\}$. For any $\alpha \in \Sigma'$ there exists a unique $r_\alpha > 0$ such that $r_\alpha \alpha \in \Sigma_{\text{red}}$. We have $\alpha \in \Sigma'^+ \cap v(\Sigma'^-)$ if and only if $r_\alpha \alpha \in \Sigma_{\text{red}}^+ \cap v(\Sigma_{\text{red}}^-)$ and $\langle \alpha, \nu(\lambda) \rangle > 0$ if and only if $\langle r_\alpha \alpha, \nu(\lambda) \rangle > 0$. Since $\alpha \mapsto r_\alpha \alpha$ gives a bijection $\Sigma' \simeq \Sigma_{\text{red}}$, we get $\#\{\alpha \in \Sigma'^+ \cap v(\Sigma'^-) \mid \langle \alpha, \nu(\lambda) \rangle > 0\} = \#\{\alpha \in \Sigma_{\text{red}}^+ \cap v(\Sigma_{\text{red}}^-) \mid \langle \alpha, \nu(\lambda) \rangle > 0\}$.

We get the first formula. Since

$$\begin{aligned} \ell(v) &= \#\{\alpha \in \Sigma_{\text{red}}^+ \mid v^{-1}(\alpha) < 0\} \\ &= \#\{\alpha \in \Sigma_{\text{red}}^+ \mid v^{-1}(\alpha) < 0, \langle \alpha, \nu(\lambda) \rangle \leq 0\} \\ &\quad + \#\{\alpha \in \Sigma_{\text{red}}^+ \mid v^{-1}(\alpha) < 0, \langle \alpha, \nu(\lambda) \rangle > 0\}, \end{aligned}$$

we get the second formula. \square

Lemma 2.17. *Let $v \in W_0$ and $\lambda \in \Lambda(1)$. We have:*

- $\ell(\lambda n_v) = \ell(\lambda) + \ell(v)$ if and only if for any $\alpha \in \Sigma^+$ such that $v^{-1}(\alpha) < 0$, $\langle \alpha, \nu(\lambda) \rangle \leq 0$.
- $\ell(\lambda n_v) = \ell(\lambda) - \ell(v)$ if and only if for any $\alpha \in \Sigma^+$ such that $v^{-1}(\alpha) < 0$, $\langle \alpha, \nu(\lambda) \rangle > 0$.
- $\ell(n_v \lambda) = \ell(\lambda) + \ell(v)$ if and only if for any $\alpha \in \Sigma^+$ such that $v(\alpha) < 0$, $\langle \alpha, \nu(\lambda) \rangle \geq 0$.
- $\ell(n_v \lambda) = \ell(\lambda) - \ell(v)$ if and only if for any $\alpha \in \Sigma^+$ such that $v(\alpha) < 0$, $\langle \alpha, \nu(\lambda) \rangle < 0$.

In particular, for $\alpha \in \Delta$ and $\lambda \in \Lambda(1)$, we have

- $\ell(\lambda n_{s_\alpha}) = \ell(\lambda) + 1$ if and only if $\langle \alpha, \nu(\lambda) \rangle \leq 0$.
- $\ell(n_{s_\alpha} \lambda) = \ell(\lambda) + 1$ if and only if $\langle \alpha, \nu(\lambda) \rangle \geq 0$.

Proof. Obvious from the previous lemma. \square

Lemma 2.18. *Let P be a parabolic subgroup, $v \in W_0$, $w \in W_P(1)$ and $\lambda_0 \in Z(W_P(1))$.*

- (1) *If $v \in W_0^P$, λ_0 is dominant and w is P -negative, then we have $\ell(n_v \lambda_0 w) = \ell(n_v \lambda_0) + \ell(w) = \ell(v) + \ell(\lambda_0) + \ell(w)$.*
- (2) *If $v \in {}^P W_0$, λ_0 is anti-dominant and w is P -positive, then we have $\ell(w \lambda_0 n_v) = \ell(w) + \ell(\lambda_0 n_v) = \ell(w) + \ell(\lambda_0) + \ell(v)$.*

Proof. (2) follows from (1) by taking the inverse.

We prove (1). Take $w_1 \in W_{0,P}$ and $\lambda \in \Lambda(1)$ such that $w = n_{w_1} \lambda$. We remark that $\nu(\lambda_0)$ and $\nu(\lambda)$ is in the same closed Weyl chamber. In fact, if $\alpha \in \Sigma^+ \setminus \Sigma_P^+$, then $\langle \alpha, \nu(\lambda_0) \rangle \geq 0$ and $\langle \alpha, \nu(\lambda) \rangle \geq 0$ by the assumption. Hence $\langle \alpha, \nu(\lambda_0) \rangle \langle \alpha, \nu(\lambda) \rangle \geq 0$. If $\alpha \in \Sigma_P^+$, then $\langle \alpha, \nu(\lambda_0) \rangle = 0$ since $\lambda_0 \in Z(W_P(1))$. Hence $\langle \alpha, \nu(\lambda) \rangle \langle \alpha, \nu(\lambda_0) \rangle \geq 0$ for any $\alpha \in \Sigma_P^+$. Therefore $\nu(\lambda)$ and $\nu(\lambda_0)$ are in the same closed Weyl chamber. In particular, $\ell(\lambda_0 \lambda) = \ell(\lambda_0) + \ell(\lambda)$ by Lemma 2.11.

Since λ_0 is in the center of $W_P(1)$, we have

$$\begin{aligned}\ell(n_v \lambda_0 w) &= \ell(n_v n_{w_1} \lambda_0 \lambda) \\ &= \ell(\lambda_0 \lambda) + \ell(v w_1) - 2\#\{\alpha \in \Sigma_{\text{red}}^+ \mid v w_1(\alpha) < 0, \langle \alpha, \nu(\lambda_0 \lambda) \rangle < 0\}\end{aligned}$$

by Lemma 2.16.

Let $\alpha \in \Sigma_{\text{red}}^+$ such that $v w_1(\alpha) < 0$. Since $v \in W_0^P$, $\ell(v w_1) = \ell(v) + \ell(w_1)$. Hence we have $w_1(\alpha) < 0$ or $v(\beta) < 0$ where $\beta = w_1(\alpha) > 0$. Assume that $v(\beta) < 0$ where $\beta = w_1(\alpha) > 0$. Since $v \in W_0^P$, $\beta \in \Sigma^+ \setminus \Sigma_P^+$. Therefore $\alpha = w_1^{-1}(\beta) \in \Sigma^+ \setminus \Sigma_P^+$. Hence $\langle \alpha, \nu(\lambda) \rangle \geq 0$ since λ is P -negative. From the assumption, λ_0 is dominant. Therefore $\langle \alpha, \nu(\lambda_0) \rangle \geq 0$. Hence we get $\langle \alpha, \nu(\lambda_0 \lambda) \rangle \geq 0$. Therefore we have

$$\ell(n_v \lambda_0 w) = \ell(\lambda_0 \lambda) + \ell(v w_1) - 2\#\{\alpha \in \Sigma_{\text{red}}^+ \mid w_1(\alpha) < 0, \langle \alpha, \nu(\lambda_0 \lambda) \rangle < 0\}.$$

Since $w_1 \in W_{0,P}$, $w_1(\alpha) < 0$ implies $\alpha \in \Sigma_P^+$. Hence $\langle \alpha, \nu(\lambda_0) \rangle = 0$. We have

$$\ell(n_v \lambda_0 w) = \ell(\lambda_0 \lambda) + \ell(v w_1) - 2\#\{\alpha \in \Sigma_{\text{red}}^+ \mid w_1(\alpha) < 0, \langle \alpha, \nu(\lambda) \rangle < 0\}.$$

Recall that we have $\ell(v w_1) = \ell(v) + \ell(w_1)$ and $\ell(\lambda_0 \lambda) = \ell(\lambda_0) + \ell(\lambda)$. Hence

$$\begin{aligned}\ell(n_v \lambda_0 w) &= \ell(\lambda_0) + \ell(v) + \ell(\lambda) + \ell(w_1) - 2\#\{\alpha \in \Sigma_{\text{red}}^+ \mid w_1(\alpha) < 0, \langle \alpha, \nu(\lambda) \rangle < 0\} \\ &= \ell(\lambda_0) + \ell(v) + \ell(n_{w_1} \lambda) = \ell(\lambda_0) + \ell(v) + \ell(w)\end{aligned}$$

by Lemma 2.16. Put $w = 1$. Then we have $\ell(n_v \lambda_0) = \ell(\lambda_0) + \ell(v)$. \square

Lemma 2.19. *Let $w \in W_P(1)$, $v \in W_0$ and $\lambda_0 \in Z(W_P(1))$.*

- (1) *If w is P -positive, $\lambda_0 = \lambda_P^+$ as in Proposition 2.5 and $v \in W_0^P$, then $\ell(n_v \lambda_0 w) = \ell(n_v \lambda_0) + \ell(w) = \ell(\lambda_0) - \ell(v) + \ell(w)$.*
- (2) *If w is P -negative, $\lambda_0 = \lambda_P^-$ as in Proposition 2.5 and $v \in {}^P W_0$, then $\ell(w \lambda_0 n_v) = \ell(w) + \ell(\lambda_0 n_v) = \ell(w) + \ell(\lambda_0) - \ell(v)$.*

Proof. (2) follows from (1) by taking the inverse.

Take $w_1 \in W_{0,P}$ and $\lambda \in \Lambda(1)$ such that $w = n_{w_1} \lambda$. Then we have

$$\begin{aligned}\ell(n_v \lambda_0 w) &= \ell(n_v n_{w_1} \lambda_0 \lambda) \\ &= \ell(\lambda_0 \lambda) - \ell(v w_1) + 2\#\{\alpha \in \Sigma_{\text{red}}^+ \mid (v w_1)(\alpha) < 0, \langle \alpha, \nu(\lambda_0 \lambda) \rangle \geq 0\}\end{aligned}$$

by Lemma 2.16. Let $\alpha \in \Sigma^+$ such that $(v w_1)(\alpha) < 0$. Then $w_1(\alpha) < 0$ or $v(\beta) < 0$ where $\beta = w_1(\alpha) > 0$. If $v(\beta) < 0$, $\beta = w_1(\alpha) > 0$, then $\beta \in \Sigma^+ \setminus \Sigma_P^+$ since $v \in W_0^P$. Since $w_1 \in W_{0,P}$, we have $\alpha = w_1^{-1}(\beta) \in \Sigma^+ \setminus \Sigma_P^+$. Hence $\langle \alpha, \nu(\lambda_0) \rangle < 0$ by the condition on λ_P^+ , $\langle \alpha, \nu(\lambda) \rangle \leq 0$ since λ is P -positive. Therefore we have $\langle \alpha, \nu(\lambda_0 \lambda) \rangle < 0$. We get

$$\ell(n_v \lambda_0 w) = \ell(\lambda_0 \lambda) - \ell(v w_1) + 2\#\{\alpha \in \Sigma_{\text{red}}^+ \mid w_1(\alpha) < 0, \langle \alpha, \nu(\lambda_0 \lambda) \rangle \geq 0\}.$$

If $w_1(\alpha) < 0$, then since $w_1 \in W_{0,P}$, we have $\alpha \in \Sigma_P$. Hence $\langle \alpha, \nu(\lambda_0) \rangle = 0$. Therefore we have

$$\ell(n_v \lambda_0 w) = \ell(\lambda_0 \lambda) - \ell(v w_1) + 2\#\{\alpha \in \Sigma_{\text{red}}^+ \mid w_1(\alpha) < 0, \langle \alpha, \nu(\lambda) \rangle \geq 0\}.$$

We have:

- $\ell(v w_1) = \ell(v) + \ell(w_1)$ since $v \in W_0^P$ and $w_1 \in W_{0,P}$.

- $\ell(\lambda_0\lambda) = \ell(\lambda_0) + \ell(\lambda)$. Indeed, $\nu(\lambda_0)$ and $\nu(\lambda)$ are in the same closed Weyl chamber. If $\alpha \in \Sigma^+ \setminus \Sigma_P^+$, then $\langle \alpha, \nu(\lambda_0) \rangle$ and $\langle \alpha, \nu(\lambda) \rangle$ are both not positive since λ_0, λ are both P -positive. If $\alpha \in \Sigma_P^+$, then $\langle \alpha, \nu(\lambda_0) \rangle = 0$, hence $\langle \alpha, \nu(\lambda_0) \rangle \langle \alpha, \nu(\lambda) \rangle = 0 \geq 0$.

Therefore we get

$$\begin{aligned} & \ell(n_v\lambda_0w) \\ &= \ell(\lambda_0) - \ell(v) + \ell(\lambda) - \ell(w_1) + 2\#\{\alpha \in \Sigma_{\text{red}}^+ \mid w_1(\alpha) < 0, \langle \alpha, \nu(\lambda) \rangle \geq 0\} \\ &= \ell(\lambda_0) - \ell(v) + \ell(\lambda n_{w_1}) = \ell(\lambda_0) - \ell(v) + \ell(w) \end{aligned}$$

by Lemma 2.16. Applying this to $w = 1$, we get $\ell(n_v\lambda_0) = \ell(\lambda_0) - \ell(v)$. \square

Lemma 2.20. *Let $w \in W_P(1)$ and $\lambda_P^- \in \Lambda(1)$ as in Proposition 2.5. Then $\ell(w\lambda_P^-) = \ell(w) + \ell(\lambda_P^-)$ if and only if w is P -negative.*

Proof. The “if part” follows from Lemma 2.19 (2). Assume that $\ell(w\lambda_P^-) = \ell(w) + \ell(\lambda_P^-)$. Take $v \in W_{0,P}$ and $\mu \in \Lambda(1)$ such that $w = n_v\mu$. Then we have

$$\ell(w\lambda_P^-) = \sum_{\alpha \in \Sigma'^+ \cap v^{-1}(\Sigma'^+)} |\langle \alpha, \nu(\mu\lambda_P^-) \rangle| + \sum_{\alpha \in \Sigma'^+ \cap v^{-1}(\Sigma'^-)} |\langle \alpha, \nu(\mu\lambda_P^-) \rangle + 1|$$

and

$$\begin{aligned} \ell(w) &= \sum_{\alpha \in \Sigma'^+ \cap v^{-1}(\Sigma'^+)} |\langle \alpha, \nu(\mu) \rangle| + \sum_{\alpha \in \Sigma'^+ \cap v^{-1}(\Sigma'^-)} |\langle \alpha, \nu(\mu) \rangle + 1|, \\ \ell(\lambda_P^-) &= \sum_{\alpha \in \Sigma'^+ \cap v^{-1}(\Sigma'^+)} |\langle \alpha, \nu(\lambda_P^-) \rangle| + \sum_{\alpha \in \Sigma'^+ \cap v^{-1}(\Sigma'^-)} |\langle \alpha, \nu(\lambda_P^-) \rangle| \end{aligned}$$

by the length formula. By the triangle inequality and the assumption $\ell(w\lambda_P^-) = \ell(w) + \ell(\lambda_P^-)$, we have

$$|\langle \alpha, \nu(\mu\lambda_P^-) \rangle + \varepsilon| = |\langle \alpha, \nu(\mu) \rangle + \varepsilon| + |\langle \alpha, \nu(\lambda_P^-) \rangle|.$$

where $\varepsilon = 1$ if $\alpha \in \Sigma'^+ \cap v^{-1}(\Sigma'^-)$ and $\varepsilon = 0$ if $\alpha \in \Sigma'^+ \cap v^{-1}(\Sigma'^+)$. If $\alpha \in \Sigma^+ \setminus \Sigma_P^+$, we have $v(\alpha) > 0$ since $v \in W_{0,P}$. Hence $\varepsilon = 0$. Therefore we get

$$|\langle \alpha, \nu(\mu\lambda_P^-) \rangle| = |\langle \alpha, \nu(\mu) \rangle| + |\langle \alpha, \nu(\lambda_P^-) \rangle|,$$

So we get $\langle \alpha, \nu(\mu) \rangle \langle \alpha, \nu(\lambda_P^-) \rangle \geq 0$. We have $\langle \alpha, \nu(\lambda_P^-) \rangle > 0$ by the condition on λ_P^- . Hence $\langle \alpha, \nu(\mu) \rangle \geq 0$. Therefore $w = n_v\mu$ is P -negative. \square

2.11. Twist by $n_{w_G w_P}$. For a parabolic subgroup P , let w_P be the longest element in $W_{0,P}$. In particular, w_G is the longest element in W_0 . Let P' be a parabolic subgroup corresponding to $-w_G(\Delta_P)$, in other words, $P' = n_{w_G w_P} P^{\text{op}} n_{w_G w_P}^{-1}$ where P^{op} is the opposite parabolic subgroup of P with respect to the Levi part of P containing Z . Set $n = n_{w_G w_P}$. Then the map $P^{\text{op}} \rightarrow P'$ defined by $p \mapsto npn^{-1}$ is an isomorphism which preserves the data used to define the pro- p -Iwahori Hecke algebras. Hence $T_w^P \mapsto T_{nwn^{-1}}^{P'}$ gives an isomorphism $\mathcal{H}_P \rightarrow \mathcal{H}_{P'}$. This sends T_w^{P*} to $T_{nwn^{-1}}^{P'*}$ and $E_{o_+, P \cdot v}^P(w)$ to $E_{o_+, P' \cdot nvn^{-1}}^{P'}(nwn^{-1})$ where $v \in W_{0,P}$.

Let σ be an \mathcal{H}_P -module. Then we define an $\mathcal{H}_{P'}$ -module $n_{w_G w_P} \sigma$ via the pull-back of the above isomorphism. Namely, we define $(n_{w_G w_P} \sigma)(T_w^{P'}) = \sigma(T_{n_{w_G w_P} w n_{w_G w_P}}^{-1})$.

Using this twist, we have another description of I_P .

Proposition 2.21 ([Vig15b, Theorem 1.8]). *Let P be a parabolic subgroup and σ an \mathcal{H}_P -module. Set $P' = n_{w_G w_P} P^{\text{op}} n_{w_G w_P}^{-1}$. Then the map $\varphi \mapsto \varphi(T_{n_w})$ gives an isomorphism*

$$\{\varphi \in I_P(\sigma) \mid \varphi(T_{n_w}) = 0 \text{ for any } w \in W_0^P \setminus \{w_G w_P\}\} \simeq n_{w_G w_P} \sigma|_{\mathcal{H}_{P'}^+},$$

as $j_{P'}^+(\mathcal{H}_{P'}^+)$ -modules. Let $x \mapsto \varphi_x$ be the inverse of the above isomorphism. Then the induced homomorphism

$$n_{w_G w_P} \sigma \otimes_{(\mathcal{H}_{P'}^+, j_{P'}^+)} \mathcal{H} \rightarrow I_P(\sigma)$$

given by $x \otimes X \mapsto \varphi_x X$ is an isomorphism.

Lemma 2.22. (1) *The map $w \mapsto w_G w_P w^{-1}$ gives a bijection $W_0^P \simeq {}^{P'} W_0$ which reverse the Bruhat order.*

(2) *The isomorphism $I_P(\sigma) \simeq n_{w_G w_P} \sigma \otimes_{(\mathcal{H}_{P'}^+, j_{P'}^+)} \mathcal{H}$ is given by $I_P(\sigma) \ni \varphi \mapsto \sum_{w \in W_0^P} \varphi(T_{n_w}) \otimes T_{n_{w_G w_P w^{-1}}}^*$.*

Proof. The first part is proved in the proof of [Abe, Proposition 4.15]. For (2), we prove the following. Let $x \in \sigma$ and $\varphi \in I_P(\sigma)$ such that $\varphi(T_{n_{w_G w_P}}) = x$ and $\varphi(T_{n_w}) = 0$ for any $w \in W_0^P \setminus \{w_G w_P\}$. Then for $v \in {}^{P'} W_0$ and $w \in W_0^P$, we have

$$(\varphi T_{n_v}^*)(T_{n_w}) = \begin{cases} x & v = w_G w_P w^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

This means the following diagram is commutative.

$$\begin{array}{ccc} I_P(\sigma) & \xleftarrow{\sim} & n_{w_G w_P} \sigma \otimes_{(\mathcal{H}_{P'}^+, j_{P'}^+)} \mathcal{H} \\ \downarrow \wr & & \downarrow \wr \\ \bigoplus_{w \in W_0^P} \sigma & \longrightarrow & \bigoplus_{v \in {}^{P'} W_0} \sigma \otimes T_{n_v}^* \\ \Psi & & \Psi \\ (x_w) & \longmapsto & (x_{v^{-1} w_G w_P} \otimes T_{n_v}^*). \end{array}$$

The commutativity of this diagram implies the lemma.

Assume that $(\varphi T_{n_v}^*)(T_{n_w}) \neq 0$. We have

$$T_{n_v}^* T_{n_w} = E_{v^{-1} \cdot o_-}(n_v) E_{o_-}(n_w) \in C E_{v^{-1} \cdot o_-}(n_v n_w) \subset \sum_{a \leq vw} T_{n_a} C[Z_\kappa].$$

For $a \in W_0$ and $t \in Z_\kappa$, we have $\varphi(T_{n_a} T_t) = \varphi(T_{n_a}) T_t^P$. Hence $\varphi(T_{n_a}) \neq 0$ for some $a \leq vw$.

Decompose $a = a_1 a_2$ where $a_1 \in W_0^P$ and $a_2 \in W_{0,P}$. Then we have $\varphi(T_{n_a}) = \varphi(T_{n_{a_1}}) T_{n_{a_2}}^P$. Since this is non-zero, we have $a_1 = w_G w_P$. Namely we have $a \in w_G W_{0,P}$. Take $b \in W_{0,P}$ such that $a = w_G b$. By (1), we can take $v_1 \in W_0^P$ such that $v = w_G w_P v_1^{-1}$. Then $a \leq vw$ implies $b \geq w_P v_1^{-1} w$. Since

$b \in W_{0,P}$, we also have $w_P v_1^{-1} w \in W_{0,P}$. Hence $v_1^{-1} w \in W_{0,P}$. Therefore we have $w \in v_1 W_{0,P}$. Since $v_1, w \in W_0^P$, we have $v_1 = w$. Hence $v = w_G w_P w^{-1}$.

If $v = w_G w_P w^{-1}$, then $\ell(v) = \ell(w_G) - \ell(w_P w^{-1}) = \ell(w_G) - \ell(w_P) - \ell(w)$ since $w \in W_0^P$. Hence $\ell(v) + \ell(w) = \ell(w_G) - \ell(w_P) = \ell(w_G w_P) = \ell(vw)$. Therefore we have

$$\begin{aligned} T_{n_v}^* T_{n_w} &= E_{o_- \cdot v^{-1}}(n_v) E_{o_-}(n_w) \\ &= E_{o_- \cdot v^{-1}}(n_v n_w) \in T_{n_{w_G w_P}} + \sum_{a < w_G w_P} T_{n_a} C[Z_\kappa]. \end{aligned}$$

By [Abe, Lemma 4.13], if $a < w_G w_P$ then $a \notin w_G w_P W_{0,P}$. Hence we have $\varphi(T_{n_a} C[Z_\kappa]) = 0$. Therefore we have $\varphi(T_{n_v}^* T_{n_w}) = \varphi(T_{n_{w_G w_P}}) = x$. \square

2.12. The extension and the generalized Steinberg modules. Let P be a parabolic subgroup and σ an \mathcal{H}_P -module. For $\alpha \in \Delta$, let P_α be a parabolic subgroup corresponding to $\{\alpha\}$. Then we define $\Delta(\sigma) \subset \Delta$ by

$$\begin{aligned} \Delta(\sigma) &= \{\alpha \in \Delta \mid \langle \Delta_P, \alpha^\vee \rangle = 0, \sigma(T_\lambda^P) = 1 \text{ for any } \lambda \in W_{\text{aff}, P_\alpha}(1) \cap \Lambda(1)\} \cup \Delta_P. \end{aligned}$$

Let $P(\sigma)$ be a parabolic subgroup corresponding to $\Delta(\sigma)$.

Proposition 2.23 ([AHVa]). *Let σ be an \mathcal{H}_P -module and Q a parabolic subgroup between P and $P(\sigma)$. Denote the parabolic subgroup corresponding to $\Delta_Q \setminus \Delta_P$ by P_2 . Then there exist a unique \mathcal{H}_Q -module $e_Q(\sigma)$ acting on the same space as σ such that*

- $e_Q(\sigma)(T_w^{Q*}) = \sigma(T_w^{P*})$ for any $w \in W_P(1)$.
- $e_Q(\sigma)(T_w^{Q*}) = 1$ for any $w \in W_{P_2, \text{aff}}(1)$.

Moreover, one of the following condition gives a characterization of $e_Q(\sigma)$.

- (1) For any $w \in W_P^{Q-}(1)$, $e_Q(\sigma)(T_w^{Q*}) = \sigma(T_w^{P*})$ (namely, $e_Q(\sigma) \simeq \sigma$ as $(\mathcal{H}_P^{Q-}, j_P^{Q-*})$ -modules) and for any $w \in W_{\text{aff}, P_2}(1)$, $e_Q(\sigma)(T_w^{Q*}) = 1$.
- (2) For any $w \in W_P^{Q+}(1)$, $e_Q(\sigma)(T_w^{Q*}) = \sigma(T_w^{P*})$ and for any $w \in W_{\text{aff}, P_2}(1)$, $e_Q(\sigma)(T_w^{Q*}) = 1$.

We call $e_Q(\sigma)$ the extension of σ to \mathcal{H}_Q . A typical example of the extension is the trivial representation $\mathbf{1} = \mathbf{1}_G$. This is a one-dimensional \mathcal{H} -module defined by $\mathbf{1}(T_w) = q_w$, or equivalently $\mathbf{1}(T_w^*) = 1$. We have $\Delta(\mathbf{1}_P) = \{\alpha \in \Delta \mid \langle \Delta_P, \alpha^\vee \rangle = 0\} \cup \Delta_P$ and, if Q is a parabolic subgroup between P and $P(\mathbf{1}_P)$, we have $e_Q(\mathbf{1}_P) = \mathbf{1}_Q$.

Remark 2.24. Assume that $p = 0$ in C . The condition “for any $w \in W_{\text{aff}, P_2}(1)$, $e_Q(\sigma)(T_w^{Q*}) = 1$ ” is equivalent to the following two conditions.

- $e_Q(\sigma)(T_s^Q) = 0$ for any $s \in S_{\text{aff}, P_2}(1) \cap W_{\text{aff}, P_2}(1)$.
- $e_Q(\sigma)(T_t^Q) = 1$ for any $t \in Z_\kappa \cap W_{\text{aff}, P_2}(1)$.

Indeed, assume that $e_Q(\sigma)(T_w^{Q*}) = 1$ for any $w \in W_{\text{aff}, P_2}(1)$. Then for $t \in Z_\kappa \cap W_{\text{aff}, P_2}(1)$, we have $T_t^{Q*} = T_t^Q$. Hence $e_Q(\sigma)(T_t^Q) = e_Q(\sigma)(T_t^{Q*}) = 1$. For $s \in S_{\text{aff}, P_2}(1) \cap W_{\text{aff}, P_2}(1)$, take $c_s(t) \in \mathbb{Z}$ as in Proposition 2.1. By Lemma 2.3, we have $e_Q(\sigma)(c_s) = \sum_{t \in Z_\kappa \cap W_{\text{aff}, P_2}(1)} c_s(t) e_Q(\sigma)(T_t^Q) = \sum_{t \in Z_\kappa \cap W_{\text{aff}, P_2}(1)} c_s(t) = q_{s, P_2} - 1 = -1$. Hence $e_Q(\sigma)(T_s^Q) = e_Q(\sigma)(T_s^{Q*}) + e_Q(\sigma)(c_s) = 1 - 1 = 0$.

On the other hand, assume that the two conditions hold. Then by the above argument, from the second condition, we have $e_Q(\sigma)(c_s) = -1$. Hence $e_Q(\sigma)(T_s^{Q*}) = e_Q(\sigma)(T_s^Q) - e_Q(\sigma)(c_s) = 1$. By taking a reduced expression of $w \in W_{\text{aff}, P_2}(1)$, we get $e_Q(\sigma)(T_w^{Q*}) = 1$. The conditions are appeared in [Abe, 4.4].

Remark 2.25. For each $\alpha \in \Delta$, let P_α be a parabolic subgroup corresponding to $\{\alpha\}$. By [Abe, Lemma 2.5], $\Lambda(1) \cap W_{\text{aff}, P_2}(1)$ is generated by $\bigcup_{\alpha \in \Delta(\sigma) \setminus \Delta_P} (\Lambda(1) \cap W_{\text{aff}, P_\alpha}(1))$. Hence for each $\lambda \in \Lambda(1) \cap W_{\text{aff}, P_2}(1)$, we can write $\lambda = \mu_1 \cdots \mu_r$ where $\mu_i \in W_{\text{aff}, P_\alpha}(1) \cap \Lambda(1)$ for some $\alpha \in \Delta(\sigma) \setminus \Delta_P$. Since $\alpha \in \Delta(\sigma) \setminus \Delta_P$ is orthogonal to Δ_P , $\ell_P(\mu_i) = 0$ for each i . Therefore $T_\lambda^P = T_{\mu_1}^P \cdots T_{\mu_r}^P$. Since $\sigma(T_{\mu_i}^P) = 1$, we have $\sigma(T_\lambda^P) = 1$.

Let $P(\sigma) \supset P_0 \supset Q_1 \supset Q \supset P$. Then as in [Abe, 4.5], we have $I_{Q_1}^{P_0}(e_{Q_1}(\sigma)) \subset I_Q^{P_0}(e_Q(\sigma))$. Define

$$\text{St}_Q^{P_0}(\sigma) = \text{Cok} \left(\bigoplus_{Q_1 \not\supset Q} I_{Q_1}^{P_0}(e_{Q_1}(\sigma)) \rightarrow I_Q^{P_0}(e_Q(\sigma)) \right).$$

When $P_0 = G$, we write $\text{St}_Q(\sigma)$.

In the rest of this subsection, we assume that $P(\sigma) = G$. Since $\Delta \setminus \Delta_P = \Delta(\sigma) \setminus \Delta_P$ is orthogonal to Δ_P , we have $w_G w_P \in W_{0, P_2}$ where P_2 corresponds to $\Delta \setminus \Delta_P$ and $n_{w_G w_P} P^{\text{op}} n_{w_G w_P}^{-1} = P$. Hence $n_{w_G w_P} \sigma$ is also an \mathcal{H}_P -module.

Lemma 2.26. $n_{w_G w_P} \sigma = \sigma$.

Proof. Let $w \in W_P(1)$. Put $n = n_{w_G w_P}$. Since $n \in W_{\text{aff}, P_2}(1)$ and $W_{\text{aff}, P_2}(1)$ is a normal subgroup of $W(1)$ [Abe, Lemma 4.17], we have $n^{-1} w n w^{-1} = n^{-1} (w n w^{-1}) \in W_{\text{aff}, P_2}(1)$. The image of n (resp. w) by $W(1) \rightarrow W \rightarrow W_0$ is in W_{0, P_2} (resp. $W_{0, P}$) and by the assumption $P(\sigma) = G$, W_{0, P_2} and $W_{0, P}$ commute with each other. Hence the image of $n^{-1} w n w^{-1}$ in W_0 is trivial. Therefore $n^{-1} w n w^{-1} \in \Lambda(1) \cap W_{\text{aff}, P_2}(1)$. In particular, the length as an element in $W_P(1)$ is zero by Lemma 2.12 and $\sigma(T_{n^{-1} w n w^{-1}}^P) = 1$ by Remark 2.25. Hence $n \sigma(T_w^P) = \sigma(T_{n^{-1} w n}^P) = \sigma(T_{n^{-1} w n w^{-1}}^P T_w^P) = \sigma(T_{n^{-1} w n w^{-1}}^P) \sigma(T_w^P) = \sigma(T_w^P)$. \square

Lemma 2.27. Let Q be a parabolic subgroup containing P and set $Q' = n_{w_G w_Q} Q^{\text{op}} n_{w_G w_Q}^{-1}$. Then we have $n_{w_G w_Q} e_Q(\sigma) \simeq e_{Q'}(\sigma)$.

Proof. By the above lemma, we have

$$\sigma|_{\mathcal{H}_P^-} = n_{w_G w_P} \sigma|_{\mathcal{H}_P^-} = n_{w_G w_Q} e_Q(n_{w_Q w_P} \sigma)|_{\mathcal{H}_P^-} = n_{w_G w_Q} e_Q(\sigma)|_{\mathcal{H}_P^-}.$$

Let Q_2 (resp. Q'_2) be the subgroup corresponding to $\Delta_Q \setminus \Delta_P$ (resp. $\Delta_{Q'} \setminus \Delta_P$). We have $w_G w_Q \in W_{0, P_2}$. Hence $n_{w_G w_Q}$ preserves Σ_{P_2} . Moreover we have $w_G w_Q(\Sigma_{Q_2}) = \Sigma_{Q'_2}$. For $\alpha \in \Sigma_{Q_2}$, the root subgroup for α is sent to that of $w_G w_Q(\alpha)$ by $n_{w_G w_Q}$. Denote the Levi part of Q_2 (resp. Q'_2) containing Z by M_{Q_2} (resp. $M_{Q'_2}$). Then the above argument implies $n_{w_G w_Q} M_{Q_2}' n_{w_G w_Q}^{-1} = M_{Q'_2}'$. Hence $n_{w_G w_Q} W_{\text{aff}, Q_2}(1) n_{w_G w_Q}^{-1} = W_{\text{aff}, Q'_2}(1)$. Therefore, for $w \in W_{\text{aff}, Q'_2}(1)$, we have

$$(n_{w_G w_Q} e_Q(\sigma))(T_w^{Q'*}) = e_Q(\sigma)(T_{n_{w_G w_Q}^{-1} w n_{w_G w_Q}}^{Q*}) = 1$$

from the definition of the extension. We get the lemma by the characterization of the extension. \square

2.13. Supersingular modules. Assume that $p = 0$ in C . Let \mathcal{O} be a conjugacy class in $W(1)$ which is contained in $\Lambda(1)$. For a spherical orientation o , set $z_{\mathcal{O}} = \sum_{\lambda \in \mathcal{O}} E_o(\lambda)$. Then this does not depend on o and gives an element of the center of \mathcal{H} [Vig15a, Theorem 5.1]. The length of $\lambda \in \mathcal{O}$ does not depend on λ . We denote it by $\ell(\mathcal{O})$.

Definition 2.28. Let π be an \mathcal{H} -module. We call π supersingular if there exists $n \in \mathbb{Z}_{>0}$ such that $\pi z_{\mathcal{O}}^n = 0$ for any \mathcal{O} such that $\ell(\mathcal{O}) > 0$.

2.14. Simple modules. Assume that C is an algebraically closed field of characteristic p . We consider the following triple (P, σ, Q) .

- P is a parabolic subgroup.
- σ is an simple supersingular \mathcal{H}_P -module.
- Q is a parabolic subgroup between P and $P(\sigma)$.

Define

$$I(P, \sigma, Q) = I_{P(\sigma)}(\text{St}_Q^{P(\sigma)}(\sigma)).$$

Theorem 2.29 ([Abe, Theorem 1.1]). *The module $I(P, \sigma, Q)$ is simple and any simple module has this form. Moreover, (P, σ, Q) is unique up to isomorphism.*

The simple supersingular modules are classified in [Oll14, Vig15a]. We do not recall the classification since we do not need it in this paper.

3. A FTILTRATION ON PARABOLIC INDUCTIONS

3.1. A filtration. Let P be a parabolic subgroup and A a subset of W_0^P . For an \mathcal{H}_P -module σ , put

$$I_P(\sigma)_A = \{\varphi \in I_P(\sigma) \mid \varphi(T_{nw}) = 0 \ (w \in W_0^P \setminus A)\}.$$

We call $A \subset W_0^P$ open if $v \in A, w \geq v$ implies $w \in A$. Assume that A is open and fix a minimal element $w \in A$. Set $A' = A \setminus \{w\}$. Then A' is also open. By Proposition 2.9, the map $I_P(\sigma)_A / I_P(\sigma)_{A'} \rightarrow \sigma$ given by $\varphi \mapsto \varphi(T_{nw})$ is a bijection. In this section, we give a description of the action of $E_{o_-}(\lambda)$ on $I_P(\sigma)_A / I_P(\sigma)_{A'}$. We start with the following lemma.

Lemma 3.1. *Let $w \in W_P(1)$ and $\lambda_0 = \lambda_P^- \in \Lambda(1)$ as in Proposition 2.5 such that $w\lambda_0$ is P -negative. Then $q_w^{1/2} q_{\lambda_0}^{1/2} q_{w\lambda_0}^{-1/2}$ does not depend on a choice of λ_0 .*

Proof. Let λ'_0 be another choice and put $\lambda_1 = \lambda_0 \lambda'_0$. Since $\nu(\lambda_0)$ and $\nu(\lambda'_0)$ belong to the same closed Weyl chamber, we have $\ell(\lambda_0 \lambda'_0) = \ell(\lambda_0) + \ell(\lambda'_0)$ by Lemma 2.11. Hence $q_{\lambda_1} = q_{\lambda_0} q_{\lambda'_0}$. By Lemma 2.19 (2), we have $\ell(w\lambda_1) = \ell(w\lambda_0) + \ell(\lambda'_0)$. Hence $q_{w\lambda_1} = q_{w\lambda_0} q_{\lambda'_0}$. Therefore we get $q_w^{1/2} q_{\lambda_0}^{1/2} q_{w\lambda_0}^{-1/2} = q_w^{1/2} q_{\lambda_1}^{1/2} q_{w\lambda_1}^{-1/2}$. Replacing λ_0 with λ'_0 , we also have $q_w^{1/2} q_{\lambda'_0}^{1/2} q_{w\lambda'_0}^{-1/2} = q_w^{1/2} q_{\lambda_1}^{1/2} q_{w\lambda_1}^{-1/2}$. We get the lemma. \square

We denote $q_w^{1/2} q_{\lambda_0}^{1/2} q_{w\lambda_0}^{-1/2}$ by $q(P, w)$. By Lemma 2.20, we have $q(P, w) = 1$ if and only if w is P -negative.

Proposition 3.2. *The subspace $I_P(\sigma)_A$ is \mathcal{A}_{o_-} -stable and the action of $E_{o_-}(\lambda)$ on $I_P(\sigma)_A/I_P(\sigma)_{A'} \simeq \sigma$ is given by $q(P, n_w^{-1} \cdot \lambda) E_{o_-, P}^P(n_w^{-1} \cdot \lambda)$.*

We need the following lemma. Recall that we have another basis $\{E_-(w) \mid w \in W(1)\}$ defined by $E_-(n_v \mu) = q_{n_v}^{-1/2} q_\mu^{-1/2} q_{n_v \mu}^{1/2} T_{n_v}^* E_{o_-}(\mu)$ for $v \in W_0$ and $\mu \in \Lambda(1)$. From the definition, we have

$$E_-(w) E_{o_-}(\lambda) = q_w^{1/2} q_\lambda^{1/2} q_{w\lambda}^{-1/2} E_-(w\lambda).$$

Lemma 3.3. *Let $X \in \mathcal{H}$, $\varphi \in I_P(\sigma)$ and $w \in W_P(1)$. Then we have $\varphi(X E_-(w)) = q(P, w) \varphi(X) \sigma(E_-^P(w))$.*

Proof. Replacing φ with φX , we may assume $X = 1$. If w is P -negative, then this follows from $q(P, w) = 1$ and $j_P^{-*}(E_-^P(w)) = E_-(w)$ [Abe, Lemma 4.6]. In general, let $\lambda_P^- \in \Lambda(1)$ as in Proposition 2.5 such that $w\lambda_P^-$ is P -negative. Then we have

$$E_-(w) E_{o_-}(\lambda_P^-) = q(P, w) E_-(w\lambda_P^-).$$

Hence we have

$$\begin{aligned} \varphi(E_-(w)) &= \varphi(E_-(w) E_{o_-}(\lambda_P^-)) \sigma(E_{o_-, P}^P(\lambda_P^-)^{-1}) \\ &= q(P, w) \varphi(E_-(w\lambda_P^-)) \sigma(E_{o_-, P}^P(\lambda_P^-)^{-1}) \\ &= q(P, w) \varphi(1) \sigma(E_-^P(w\lambda_P^-) E_{o_-, P}^P(\lambda_P^-)^{-1}) \\ &= q(P, w) \varphi(1) \sigma(E_-^P(w)) \end{aligned}$$

We get the lemma. \square

We also use:

Lemma 3.4. *Let $v \in W_0^P$ and $\varphi \in I_P(\sigma)$. Assume that $\varphi(T_{n_v}^*) = 0$. Then we have $\varphi(E_-(n_v w)) = 0$ for any $w \in W_P(1)$.*

Proof. Take λ_P^- as in Proposition 2.5 such that $w\lambda_P^-$ is P -negative. Then we have

$$\begin{aligned} \varphi(E_-(n_v w)) &= \varphi(E_-(n_v w) E_{o_-}(\lambda_P^-)) \sigma(E_{o_-, P}^P(\lambda_P^-)^{-1}) \\ &\in C[q_s] \varphi(E_-(n_v w\lambda_P^-)) \sigma(E_{o_-, P}^P(\lambda_P^-)^{-1}). \end{aligned}$$

Hence it is sufficient to prove $\varphi(E_-(n_v w\lambda_P^-)) = 0$. Namely we may assume w is P -negative.

If w is P -negative, by Lemma 2.18, we have $\ell(n_v w) = \ell(n_v) + \ell(w)$. Hence by the definition of E_- , we have $E_-(n_v w) = T_{n_v}^* E_-(w)$. Therefore we have $\varphi(E_-(n_v w)) = \varphi(T_{n_v}^*) \sigma(E_-^P(w)) = 0$. \square

Proof of Proposition 3.2. Let $v \notin A'$. By the Bernstein relations [Vig16, Corollary 5.43], in $\mathcal{H}[q_s^{\pm 1}]$, we have

$$E_{o_-}(\lambda) T_{n_v} \in T_{n_v} E_{o_-}(n_v^{-1} \cdot \lambda) + \sum_{v_1 < v, \mu \in \Lambda(1)} C[q_s^{\pm 1}] T_{n_{v_1}} E(\mu).$$

Since $T_{n_{v_1}} \in \sum_{v_2 \leq v_1} T_{n_{v_2}}^* C[Z_\kappa]$, we have

$$\begin{aligned} E_{o_-}(\lambda)T_{n_v} &\in T_{n_v}E_{o_-}(n_v^{-1} \cdot \lambda) + \sum_{v_2 < v, \mu \in \Lambda(1)} C[q_s^{\pm 1}]T_{n_{v_2}}^* E(\mu) \\ &= T_{n_v}E_{o_-}(n_v^{-1} \cdot \lambda) + \sum_{v_2 < v, \mu \in \Lambda(1)} C[q_s^{\pm 1}]E_-(n_{v_2}\mu). \end{aligned}$$

For $v_2 < v$, take $v_3 \in W_0^P$ and $v'_3 \in W_{0,P}$ such that $v_2 = v_3v'_3$. Then $E_-(n_{v_2}\mu) = E_-(n_{v_3}n_{v'_3}\mu)$ and $n_{v'_3}\mu \in W_P(1)$. We have $v_3 \leq v_2 < v$. Hence

$$\begin{aligned} E_{o_-}(\lambda)T_{n_v} &\in T_{n_v}E_{o_-}(n_v^{-1} \cdot \lambda) + \left(\sum_{v_3 < v, v_3 \in W_0^P, x \in W_P(1)} C[q_s^{\pm 1}]E_-(n_{v_3}x) \cap \mathcal{H} \right) \\ &= T_{n_v}E_{o_-}(n_v^{-1} \cdot \lambda) + \sum_{v_3 < v, v_3 \in W_0^P, x \in W_P(1)} C[q_s]E_-(n_{v_3}x). \end{aligned}$$

Let $\varphi \in I_P(\sigma)_A$ and we prove $\varphi(E_-(n_{v_3}x)) = 0$ for $v_3 < v, v_3 \in W_0^P, x \in W_P(1)$ by applying the above lemma. We check $\varphi(T_{n_{v_3}}^*) = 0$. We have $T_{n_{v_3}}^* \in \sum_{v_4 \leq v_3} T_{n_{v_4}} C[Z_\kappa]$. Since $v_4 \leq v_3 < v$ and $v \notin A'$, we have $v_4 \notin A$. Hence $\varphi(T_{n_{v_4}}) = 0$. Therefore we get $\varphi(T_{n_{v_3}}^*) = 0$.

Therefore we have $(\varphi E_{o_-}(\lambda))(T_{n_v}) = \varphi(T_{n_v}E_{o_-}(n_v^{-1} \cdot \lambda))$. By Lemma 3.3, we have $\varphi(T_{n_v}E_{o_-}(n_v^{-1} \cdot \lambda)) = q(P, n_v^{-1} \cdot \lambda)\varphi(T_{n_v})\sigma(E_{o_-}^P(n_v^{-1} \cdot \lambda))$. This is zero if $v \neq w$. Hence $\varphi E_{o_-}(\lambda) \in I_P(\sigma)_A$. If $v = w$, we get $\varphi(E_{o_-}(\lambda)T_{n_w}) = q(P, n_w^{-1} \cdot \lambda)\varphi(T_{n_w})\sigma(E_{o_-}^P(n_w^{-1} \cdot \lambda))$. This gives the lemma. \square

Finally, we describe the filtration in terms of a tensor product. Recall that we have an isomorphism (Proposition 2.21)

$$I_P(\sigma) \simeq n_{w_G w_P} \sigma \otimes_{(\mathcal{H}_{P'}^+, j_{P'}^+)} \mathcal{H}$$

where $P' = n_{w_G w_P} P^{\text{op}} n_{w_G w_P}^{-1}$. Let $A \subset {}^{P'}W_0$ be a closed subset, namely a subset which satisfies that $v \in A, w \leq v$ implies $w \in A$. Set $A_0 = \{w^{-1}w_G w_P \mid w \in A\}$. Then $A_0 \subset W_0^P$ is an open subset by Lemma 2.22. By Lemma 2.22, $I_P(\sigma)_{A_0}$ corresponds to

$$\sum_{v \in A} n_{w_G w_P} \sigma \otimes T_{n_v}^*.$$

Let $w \in A$ be a maximal element and put $A' = A \setminus \{w\}$.

Lemma 3.5. *The quotient*

$$\left(\sum_{v \in A} n_{w_G w_P} \sigma \otimes T_{n_v}^* \right) / \left(\sum_{v \in A'} n_{w_G w_P} \sigma \otimes T_{n_v}^* \right)$$

is isomorphic to σ as a vector space and the action of $E_{o_-}(\lambda)$ is given by

$$q(P, n_{w^{-1}w_G w_P}^{-1} \cdot \lambda)\sigma(E_{o_-,P}^P(n_{w^{-1}w_G w_P}^{-1} \cdot \lambda)).$$

Remark 3.6. Since $\ell(w) + \ell(w^{-1}w_Gw_P) = \ell(w_Gw_P)$ (see the last part of the proof of Lemma 2.22), we have $n_{w^{-1}w_Gw_P} = n_w^{-1}n_{w_Gw_P}$. Hence we have

$$\begin{aligned} \sigma(E_{o_{-,P}}^P(n_{w^{-1}w_Gw_P} \cdot \lambda)) &= \sigma(E_{o_{-,P}}^P(n_{w_Gw_P}^{-1}n_w \cdot \lambda)) \\ &= (n_{w_Gw_P}\sigma)(E_{o_{-,P'}}^{P'}(n_w \cdot \lambda)). \end{aligned}$$

Remark 3.7. For any μ , $q(P, n_{w_Gw_P}^{-1} \cdot \mu) = 1$ if and only if $n_{w_Gw_P}^{-1} \cdot \mu$ is P -negative. Set $P' = n_{w_Gw_P} P^{\text{op}} n_{w_Gw_P}^{-1}$. Then we have $(w_Gw_P)(\Sigma^+ \setminus \Sigma_P^+) = \Sigma^- \setminus \Sigma_{P'}^-$. Hence $q(P, n_{w_Gw_P}^{-1} \cdot \mu) = 1$ if and only if μ is P' -positive. Therefore $q(P, n_{w^{-1}w_Gw_P} \cdot \lambda) = 1$ if and only if $n_w \cdot \lambda$ is P' -positive.

3.2. Sum and intersections. In this subsection, assume that P is a parabolic subgroup and σ an \mathcal{H}_P -module which has the extension to \mathcal{H} . Let Q be a parabolic subgroup containing P . Let $A \subset W_0^Q$ be an open subset. Then we have $I_Q(\sigma)_A \subset I_Q(\sigma)$. In this subsection we prove the following lemma using an argument in [AHVa].

Lemma 3.8. *Let $\mathcal{P} \subset \{Q_1 \supset Q\}$ be a subset. Then we have*

$$I_Q(e_Q(\sigma))_A \cap \sum_{Q_1 \in \mathcal{P}} (I_{Q_1}(e_{Q_1}(\sigma))) = \sum_{Q_1 \in \mathcal{P}} (I_Q(e_Q(\sigma))_A \cap I_{Q_1}(e_{Q_1}(\sigma))).$$

Remark 3.9. The above lemma is equivalent to the following. Let $\mathcal{P} \subset \{Q_1 \supset Q\}$ be a subset and set $\pi = I_Q(e_Q(\sigma)) / \sum_{Q_1 \in \mathcal{P}} I_{Q_1}(e_{Q_1}(\sigma))$. Put $I_{Q_1,A} = I_{Q_1}(e_{Q_1}(\sigma)) \cap I_Q(e_Q(\sigma))_A$ and let π_A be the image of $I_Q(e_Q(\sigma))_A$. Then the sequence

$$\bigoplus_{Q_1 \in \mathcal{P}} I_{Q_1,A} \rightarrow I_{Q,A} \rightarrow \pi_A \rightarrow 0$$

is exact.

Take a minimal element $w \in A$ and set $A' = A \setminus \{w\}$.

Lemma 3.10. *Let $Q_1 \supset Q$. The injective map $I_{Q_1,A}/I_{Q_1,A'} \hookrightarrow I_{Q,A}/I_{Q,A'}$ is surjective if $w \in W_0^{Q_1}$ and 0 otherwise.*

Proof. Recall that $\varphi \mapsto \varphi(T_{n_w})$ gives an isomorphism $I_{Q,A}/I_{Q,A'} \simeq \sigma$. Assume that $w \in W_0^{Q_1}$ and let $\varphi \in I_{Q,A}$. Set $x = \varphi(T_{n_w})$ and take $\psi \in I_{Q_1}(e_{Q_1}(\sigma))$ such that $\psi(T_{n_w}) = x$ and $\psi(T_{n_v}) = 0$ for $v \in W_0^{Q_1} \setminus \{w\}$. Let $v \in W_0^Q$ and take $v_1 \in W_0^{Q_1}$ and $v_2 \in W_{Q_1,0}$ such that $v = v_1v_2$. Then we have $\psi(T_{n_v}) = \psi(T_{n_{v_1}})e_{Q_1}(\sigma)(T_{n_{v_2}}^{Q_1})$ since $j_{Q_1}^{-*}(T_{n_{v_2}}^{Q_1}) = T_{n_{v_2}}$ by Lemma 2.7. Hence if $\psi(T_{n_v}) \neq 0$, then $v_1 = w$. Therefore $v \in wW_{Q_1,0}$. Since $w \in W_0^{Q_1}$, any element in $wW_{Q_1,0}$ is greater than or equal to w . Hence $v \in A$. Namely $\psi \in I_{Q_1,A}$ and we proved the surjectivity of the map in the lemma.

Assume that $w \notin W_0^{Q_1}$ and take $w_1 \in W_0^{Q_1}$ and $w_2 \in W_{Q_1,0}$ such that $w = w_1w_2$. Then $w_1 < w$. Since w is minimal in A , we have $w_1 \notin A$. Hence for $\psi \in I_{Q_1,A}$, we have $\psi(T_{n_{w_1}}) = 0$. Therefore we have $\psi(T_{n_w}) = \psi(T_{n_{w_1}})e_{Q_1}(\sigma)(T_{n_{w_2}}^{Q_1}) = 0$. Hence we have $\psi \in I_{Q_1,A'}$. We get $I_{Q_1,A} = I_{Q_1,A'}$. \square

Proof of Lemma 3.8. Obviously the left hand side contains the right hand side. We prove that the right hand side contains the left hand side by backward induction on $\#A$. We assume that the lemma is true for A and prove the lemma for A' . By inductive hypothesis, we have

$$\begin{aligned} I_{Q,A'} \cap \sum_{Q_1 \in \mathcal{P}} I_{Q_1}(e_{Q_1}(\sigma)) &= I_{Q,A'} \cap I_{Q,A} \cap \sum_{Q_1 \in \mathcal{P}} I_{Q_1}(e_{Q_1}(\sigma)) \\ &= I_{Q,A'} \cap \sum_{Q_1 \in \mathcal{P}} I_{Q_1,A}. \end{aligned}$$

First assume that $w \notin W_0^{Q_1}$ for any $Q_1 \in \mathcal{P}$. Then by the above lemma, we have $I_{Q_1,A} = I_{Q_1,A'}$ for any $Q_1 \in \mathcal{P}$. Hence

$$I_{Q,A'} \cap \sum_{Q_1 \in \mathcal{P}} I_{Q_1,A} = I_{Q,A'} \cap \sum_{Q_1 \in \mathcal{P}} I_{Q_1,A'} = \sum_{Q_1 \in \mathcal{P}} I_{Q_1,A'}.$$

Now assume that there exists $Q_1 \in \mathcal{P}$ such that $w \in W_0^{Q_1}$. Take $\varphi_{Q_1} \in I_{Q_1,A}$ such that $\sum_{Q_1 \in \mathcal{P}} \varphi_{Q_1} \in I_{Q,A'}$. Set $\mathcal{P}_1 = \{Q_1 \in \mathcal{P} \mid w \in W_0^{Q_1}\}$. Let Q_0 be a parabolic subgroup corresponding to $\bigcup_{Q_1 \in \mathcal{P}_1} \Delta_{Q_1}$. By the above lemma, for each $Q_1 \in \mathcal{P}_1$, we have $I_{Q_0,A}/I_{Q_0,A'} \simeq I_{Q_1,A}/I_{Q_1,A'}$. Therefore for each $Q_1 \in \mathcal{P}_1$ there exists $\varphi'_{Q_1} \in I_{Q_0,A}$ such that $\varphi_{Q_1} - \varphi'_{Q_1} \in I_{Q_1,A'}$. Then we have

$$\begin{aligned} \sum_{Q_1 \in \mathcal{P}} \varphi_{Q_1} &= \sum_{Q_1 \in \mathcal{P} \setminus \mathcal{P}_1} \varphi_{Q_1} + \sum_{Q_1 \in \mathcal{P}_1} (\varphi_{Q_1} - \varphi'_{Q_1}) + \sum_{Q_1 \in \mathcal{P}_1} \varphi'_{Q_1} \\ &\in \sum_{Q_1 \in \mathcal{P} \setminus \mathcal{P}_1} I_{Q_1,A} + \sum_{Q_1 \in \mathcal{P}} I_{Q_1,A'} + \sum_{Q_1 \in \mathcal{P}_1} \varphi'_{Q_1}. \end{aligned}$$

By the above lemma, for $Q_1 \in \mathcal{P} \setminus \mathcal{P}_1$, we have $I_{Q_1,A} = I_{Q_1,A'}$. Hence

$$\sum_{Q_1 \in \mathcal{P}} \varphi_{Q_1} \in \sum_{Q_1 \in \mathcal{P} \setminus \mathcal{P}_1} I_{Q_1,A'} + \sum_{Q_1 \in \mathcal{P}_1} I_{Q_1,A'} + \sum_{Q_1 \in \mathcal{P}_1} \varphi'_{Q_1}.$$

In particular, $\sum_{Q_1 \in \mathcal{P}_1} \varphi'_{Q_1} \in I_{Q,A'} \cap I_{Q_0}(e_{Q_0}(\sigma)) = I_{Q_0,A'}$ since $\sum_{Q_1 \in \mathcal{P}} \varphi_{Q_1} \in I_{Q,A'}$. For $Q_1 \in \mathcal{P}_1$, we have $I_{Q_0,A'} \subset I_{Q_1,A'}$. Hence $I_{Q_0,A'} \subset \sum_{Q_1 \in \mathcal{P}_1} I_{Q_1,A'}$. Therefore

$$\sum_{Q_1 \in \mathcal{P}} \varphi_{Q_1} \in \sum_{Q_1 \in \mathcal{P} \setminus \mathcal{P}_1} I_{Q_1,A'} + \sum_{Q_1 \in \mathcal{P}_1} I_{Q_1,A'} + \sum_{Q_1 \in \mathcal{P}_1} I_{Q_1,A'} = \sum_{Q_1 \in \mathcal{P}} I_{Q_1,A'}.$$

We get the lemma. \square

3.3. A filtration on generalized Steinberg modules. As in the previous section, let P be a parabolic subgroup and σ an \mathcal{H}_P -module which has the extension to \mathcal{H} . Let Q be a parabolic subgroup containing P . As in Remark 3.9, for each open subset $A \subset W_0^Q$, set $I_{Q_1,A} = I_{Q_1}(e_{Q_1}(\sigma)) \cap I_Q(e_Q(\sigma))_A$ and let $\text{St}_{Q,A}$ be the image of $I_Q(e_Q(\sigma))_A$. Let $w \in A$ be a minimal element and put $A' = A \setminus \{w\}$. Then we have a commutative

diagram.

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
\oplus_{Q_1 \supsetneq Q} I_{Q_1, A'} & \longrightarrow & I_{Q, A'} & \longrightarrow & \text{St}_{Q, A'} & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
\oplus_{Q_1 \supsetneq Q} I_{Q_1, A} & \longrightarrow & I_{Q, A} & \longrightarrow & \text{St}_{Q, A} & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
\oplus_{Q_1 \supsetneq Q} I_{Q_1, A}/I_{Q_1, A'} & \longrightarrow & I_{Q, A}/I_{Q, A'} & \longrightarrow & \text{St}_{Q, A}/\text{St}_{Q, A'} & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0. &
\end{array}$$

Since the first two rows are exact by Remark 3.9, the third row is also exact. If $w \notin W_0^{Q_1}$ for any $Q_1 \supsetneq Q$, we have $I_{Q_1, A}/I_{Q_1, A'} = 0$ for any $Q_1 \supsetneq Q$ by Lemma 3.10. Hence $I_{Q, A}/I_{Q, A'} \xrightarrow{\sim} \text{St}_{Q, A}/\text{St}_{Q, A'}$. If $w \in W_0^{Q_1}$ for some $Q_1 \supsetneq Q$, we have $I_{Q_1, A}/I_{Q_1, A'} \xrightarrow{\sim} I_{Q, A}/I_{Q, A'}$ by Lemma 3.10. Hence $\oplus_{Q_1 \supsetneq Q} I_{Q_1, A}/I_{Q_1, A'} \rightarrow I_{Q, A}/I_{Q, A'}$ is surjective. Therefore we have $\text{St}_{Q, A}/\text{St}_{Q, A'} = 0$. Summarizing this argument, we get the following lemma.

Lemma 3.11. *If $w \in W_0^{Q_1}$ for some $Q_1 \supsetneq Q$ then $\text{St}_{Q, A}/\text{St}_{Q, A'} = 0$. If $w \notin W_0^{Q_1}$ for any $Q_1 \supsetneq Q$ then $I_{Q, A}/I_{Q, A'} \xrightarrow{\sim} \text{St}_{Q, A}/\text{St}_{Q, A'}$.*

Using this, we give a description of $\text{St}_P(\sigma)$. Let P_2 be a parabolic subgroup corresponding to $\Delta \setminus \Delta_P$. Note that, since σ is assumed to be have the extension to \mathcal{H} , Δ_P and $\Delta \setminus \Delta_P = \Delta_{P_2}$ are orthogonal to each others. Hence $W_0^P = W_{0, P_2}$.

Proposition 3.12. *The representation $\pi = \text{St}_P(\sigma)$ is isomorphic to σ as an (\mathcal{H}_P^+, j_P^+) -module and for $w \in W_{\text{aff}, P_2}(1)$, $\pi(T_w) = (-1)^{\ell(w)}$.*

Proof. Let $w \in W_0^P$ and assume that $w \notin W_0^{Q_1}$ for any $Q_1 \supsetneq P$. Let $\alpha \in \Delta \setminus \Delta_P$ and consider the parabolic subgroup Q_1 corresponding to $\Delta_P \cup \{\alpha\}$. Then by the assumption $w(\Delta_{Q_1}) \not\subset \Sigma^+$. Since $w(\Delta_P) \subset \Sigma^+$, we have $w(\alpha) < 0$. Therefore $w \in W_0^P = W_{0, P_2}$ satisfies that $w(\alpha) < 0$ for any $\alpha \in \Delta_{P_2}$. Hence $w = w_{P_2} = w_G w_P$. Combining the above lemma, we get the following. Note that $\{w_G w_P\}$ is open.

- $\text{St}_P(\sigma)/\text{St}_{P, \{w_G w_P\}}$ has a filtration with zero successive quotients. Hence $\text{St}_P(\sigma) = \text{St}_{P, \{w_G w_P\}}$.
- $\text{St}_{P, \{w_G w_P\}} \simeq I_P(\sigma)_{\{w_G w_P\}}$.

Set $\sigma' = I_P(\sigma)_{\{w_G w_P\}} = \{\varphi \in I_P(\sigma) \mid \varphi(T_{n_v}) = 0 \ (v \in W_0^P \setminus \{w_G w_P\})\}$. Then $\sigma' \hookrightarrow I_P(\sigma) \rightarrow \text{St}_P(\sigma)$ is an isomorphism. By Proposition 2.21, σ' is $j_P^+(\mathcal{H}_P^+)$ -stable and isomorphic to $n_{w_G w_P} \sigma$ as (\mathcal{H}_P^+, j_P^+) -modules. By Lemma 2.26, we get the first part of the proposition.

Next, we prove $\pi(T_w) = (-1)^{\ell(w)}$ for $w \in W_{\text{aff}, P_2}(1)$ by the following three steps.

- (1) The claim is true for $w = n_v$ where $v \in W_{0,P_2}$.
- (2) For any $v_1, v_2 \in W_{0,P_2}$ and $w \in W(1)$, we have $\pi(T_{n_{v_1}wn_{v_2}}) = \pi(T_{n_{v_1}})\pi(T_w)\pi(T_{n_{v_2}})$.
- (3) We have $\pi(T_w) = (-1)^{\ell(w)}$ for any $w \in W_{\text{aff},P_2}(1)$.
- (1) We may assume $v = s \in S_{0,P_2}$. Let $\overline{\varphi} \in \text{St}_P(\sigma)$ and $\varphi \in \sigma'$ its lift. For $w \in W_0^P = W_{0,P_2}$, we have

$$(\varphi T_{n_s})(T_{n_w}) = \begin{cases} \varphi(T_{n_{wP_2}}) & w = sw_{P_2}, \\ (q_s - 1)\varphi(T_{n_{wP_2}}) & w = w_{P_2}, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, if $w < sw$, then $(\varphi T_{n_s})(T_{n_w}) = \varphi(T_{n_{sw}})$. Hence if $sw \neq w_{P_2}$ then this is zero. If $w > sw$, then $(\varphi T_{n_s})(T_{n_w}) = \varphi(T_{n_s}^2 T_{n_{sw}}) = \varphi(c_{n_s} T_{n_w} + q_s T_{n_{sw}})$. Since $s(sw) > sw$, $sw \neq w_{P_2}$. Hence $\varphi(T_{n_{sw}}) = 0$. Therefore $(\varphi T_{n_s})(T_{n_w}) = \varphi(c_{n_s} T_{n_w}) = \varphi(T_{n_w})e_Q(\sigma)(n_w^{-1} \cdot c_{n_s})$. This is zero if $w \neq w_{P_2}$. When $w = w_{P_2}$, take $c_t \in \mathbb{Z}$ such that $n_{w_{P_2}}^{-1} \cdot c_{n_s} = \sum_t c_t T_t$. Since $n_{w_{P_2}}^{-1} \cdot c_{n_s} \in C[Z_\kappa \cap W_{\text{aff},P_2}(1)]$ (Lemma 2.3), $c_t \neq 0$ implies $t \in Z_\kappa \cap W_{\text{aff},P_2}(1)$. Hence $e_Q(\sigma)(T_t) = 1$ for such t . Therefore $e_Q(\sigma)(n_{w_{P_2}}^{-1} \cdot c_{n_s}) = \sum_t c_t = q_s - 1$ by Proposition 2.1. We get the above calculation.

Take $\alpha \in \Delta_{P_2}$ such that $s = s_\alpha$ and put $\alpha' = -w_{P_2}(\alpha)$ and $s' = s_{\alpha'}$. Let Q_1 be a parabolic subgroup corresponding to $\Delta_P \cup \{\alpha'\}$. Then $w_{Q_1} = w_P s'$ and $w_{Gw_{Q_1}} = w_{P_2} s'$. Define $\psi \in I_{Q_1}(e_{Q_1}(\sigma))$ by $\psi(T_{n_w}) = 0$ for $w \in W_0^{Q_1} \setminus \{w_{P_2} s'\}$ and $\psi(T_{n_{w_{P_2} s'}}) = \varphi(T_{n_{w_{P_2}}})$. We prove

$$\psi(T_{n_w}) = \begin{cases} \varphi(T_{n_{wP_2}}) & w = sw_{P_2}, \\ q_s \varphi(T_{n_{wP_2}}) & w = w_{P_2}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $w \in W_0^P = W_{0,P_2}$ and take $w_1 \in W_0^{Q_1}$ and $w_2 \in W_{0,Q_1 \cap P_2}$ such that $w = w_1 w_2$. Then we have $\psi(T_{n_w}) = \psi(T_{n_{w_1}})e_{Q_1}(T_{n_{w_2}}^{Q_1})$. Hence if $w_1 \neq w_{P_2} s'$, namely $w \notin w_{P_2} s' W_{0,Q_1 \cap P_2}$, we have $\psi(T_{n_w}) = 0$. Note that $W_{0,Q_1 \cap P_2} = \{1, s'\}$ and $w_{P_2} s' = sw_{P_2}$. Therefore $\psi(T_{n_w}) = 0$ if $w \neq w_{P_2}, sw_{P_2}$. We have $\psi(T_{n_{w_{P_2}}}) = \psi(T_{n_{w_{P_2} s'}})e_{Q_1}(\sigma)(T_{n_{s'}}^{Q_1})$ and $e_{Q_1}(\sigma)(T_{n_{s'}}^{Q_1}) = e_{Q_1}(\sigma)(T_{n_{s'}}^{Q_1*} + c_{n_{s'}}) = 1 + q_{s'} - 1 = q_{s'}$. Since s and s' is conjugate, $q_s = q_{s'}$. Hence $\psi(T_{n_{w_{P_2}}}) = q_s \varphi(T_{n_{w_{P_2}}})$. Finally we have $\psi(T_{n_{sw_{P_2}}}) = \varphi(T_{n_{w_{P_2}}})$ by the definition of ψ . We get the above formula.

Therefore $(\varphi T_{n_s} - \psi)(T_{n_w}) = 0 = -\varphi(T_{n_w})$ if $w \neq w_{P_2}$ and equal to $-\varphi(T_{n_{w_{P_2}}})$ if $w = w_{P_2}$. Since an element in $I_P(\sigma)$ is determined by the value at $W_0^P = W_{0,P_2}$, we have $\varphi T_{n_s} - \psi = -\varphi$. Hence $\overline{\varphi} T_{n_s} = -\overline{\varphi}$ in $\text{St}_P(\sigma)$.

(2) Assume that $v_2 = 1$. To prove (2), by induction on the length of v_1 , we may assume $v_1 = s \in S_0$. If $n_s w > w$, then it is obvious. Assume that $n_s w < w$. Then we have $\pi(T_w) = \pi(T_{n_s^{-1}} T_{n_{sw}}) = \pi(T_{n_s^{-1}}) \pi(T_{n_{sw}}) = -\pi(T_{n_{sw}})$ by (1). Hence $\pi(T_{n_{sw}}) = -\pi(T_w) = \pi(T_{n_s}) \pi(T_w)$. The same argument implies (2) for any v_2 .

(3) Take $w \in W_{\text{aff},P_2}(1)$. Then we can take $w_1, w_2 \in W_{0,P_2}$ and $\lambda \in \Lambda(1) \cap W_{\text{aff},P_2}(1)$ such that $w = n_{w_1} \lambda n_{w_2}$ and λ is anti-dominant with respect

to $\Sigma_{P_2}^+$. Since Δ_P is orthogonal to Δ_{P_2} , $\Sigma_{P_2}^+ = \Sigma^+ \setminus \Sigma_P^+$. Hence λ is P -positive. As in Remark 2.25, $\sigma(T_\lambda^P) = 1$.

Since $\lambda \in \Lambda(1) \cap W_{\text{aff}}(1)$, $\ell(\lambda)$ is even by Lemma 2.14. On the other hand, since λ is P -positive, we have $\pi(T_\lambda) = \sigma(T_\lambda^P)$. Therefore $\pi(T_\lambda) = 1 = (-1)^{\ell(\lambda)}$. Using (2), we have

$$\pi(T_w) = \pi(T_{n_{w_1}})\pi(T_\lambda)\pi(T_{n_{w_2}}) = (-1)^{\ell(w_1)+\ell(\lambda)+\ell(w_2)} = (-1)^{\ell(w)}.$$

We get the proposition. \square

3.4. Tensor product decomposition. For the content of this subsection, see also [AHVa]. We start with the following lemma.

Lemma 3.13. *Let P_1, P_2 be parabolic subgroups such that Δ_{P_1} is orthogonal to Δ_{P_2} and $\Delta_{P_1} \cup \Delta_{P_2} = \Delta$. Assume that an \mathcal{H}_{P_i} -module σ_i has the extension to \mathcal{H} . Then we have the unique action of \mathcal{H} on $e_G(\sigma_1) \otimes e_G(\sigma_2)$ which satisfies $(x_1 \otimes x_2)T_w^* = x_1T_w^* \otimes x_2T_w^*$ for $w \in W(1)$, $x_1 \in e_G(\sigma_1)$ and $x_2 \in e_G(\sigma_2)$.*

Proof. The action obviously satisfies the braid relations. It is sufficient to prove that the action satisfies the quadratic relations.

Let $s \in W_{\text{aff}}(1)$ be a lift of a simple reflection. If $s \in W_{P_1}(1)$, then $s \in W_{\text{aff}, P_1}(1)$. Hence $e_G(\sigma_2)(T_s^*) = 1$. Therefore $x_1T_s^* \otimes x_2T_s^* = x_1T_s^* \otimes x_2$. Hence it satisfies the quadratic relations. The quadratic relations hold too if $s \in W_{P_2}(1)$. \square

Remark 3.14. By the above proof, we have $(x_1 \otimes x_2)T_w^* = x_1T_w^* \otimes x_2$ if $w \in W_{P_1, \text{aff}}(1)$. Hence $(x_1 \otimes x_2)X = x_1X \otimes x_2$ for any $X \in \bigoplus_{w \in W_{P_1, \text{aff}}(1)} CT_w^*$. In particular, for $X = T_w$ or $X = E_o(w)$ for any $w \in W_{P_1, \text{aff}}(1)$ and any orientation o . We also have $(x_1 \otimes x_2)X = x_1 \otimes x_2X$ for any $X = T_w$ or $X = E_o(w)$ where $w \in W_{P_2, \text{aff}}(1)$.

In the rest of this subsection, let P be a parabolic subgroup, σ an \mathcal{H}_P -module which has the extension to \mathcal{H} . Take a parabolic subgroup Q containing P . Let P_2 be the parabolic subgroup corresponding to $\Delta \setminus \Delta_P$. Note that Δ_{P_2} is orthogonal to Δ_P .

Lemma 3.15. *We have $I_Q(1) \simeq e_G(I_{P \cap Q}^{P_2}(1))$. More generally, for an $\mathcal{H}_{P_2 \cap Q}$ -module σ which has an extension to \mathcal{H}_Q , we have $I_Q(e_Q(\sigma)) \simeq e_G(I_{P_2 \cap Q}^{P_2}(\sigma))$.*

We use the following lemma.

Lemma 3.16. *Let P, Q be parabolic subgroups.*

- (1) *Let λ_P^- as in Proposition 2.5. Then $\mathcal{H}_{P \cap Q}^{P_-} \simeq \mathcal{H}_{P \cap Q}^-(T_{\lambda_P^-}^{P \cap Q})^{-1}$.*
- (2) *Let λ_P^+ as in Proposition 2.5. Then $\mathcal{H}_{P \cap Q}^{P_+} \simeq \mathcal{H}_{P \cap Q}^+(T_{\lambda_P^+}^{P \cap Q})^{-1}$.*

Proof. We only prove the first statement. The proof of the second statement is the same.

Let $\lambda \in \Lambda(1)$ such that $\langle \alpha, \nu(\lambda) \rangle \geq 0$ for any $\alpha \in \Sigma_P^+ \setminus \Sigma_{P \cap Q}^+$. We can take $n \in \mathbb{Z}_{>0}$ such that $\langle \alpha, \nu(\lambda(\lambda_P^-)^n) \rangle \geq 0$ for any $\alpha \in \Sigma^+ \setminus \Sigma_P^+$. Since $\langle \alpha, \nu((\lambda_P^-)^n) \rangle = 0$ for any $\alpha \in \Sigma_P^+$, we have $\langle \alpha, \nu(\lambda(\lambda_P^-)^n) \rangle = \langle \alpha, \nu(\lambda) \rangle \geq 0$ for any $\alpha \in \Sigma_P^+ \setminus \Sigma_{P \cap Q}^+$. Hence $\langle \alpha, \nu(\lambda(\lambda_P^-)^n) \rangle \geq 0$ for any $\alpha \in \Sigma^+ \setminus \Sigma_{P \cap Q}^+$.

Therefore for any $w \in W_{P \cap Q}(1)$ which is $(P \cap Q)$ -negative in P , there exists $n \in \mathbb{Z}_{\geq 0}$ such that $w(\lambda_P^-)^n$ is $(P \cap Q)$ -negative in G . Then we have $E_{o_-, P \cap Q}^{P \cap Q}(w) = E_{o_-, P \cap Q}^{P \cap Q}(w(\lambda_P^-)^n)(T_{\lambda_P^-}^{P \cap Q})^{-n} \in \mathcal{H}_{P \cap Q}^-(T_{\lambda_P^-}^{P \cap Q})^{-1}$. \square

Proof of Lemma 3.15. Consider the map

$$I_Q(e_Q(\sigma)) = \text{Hom}_{(\mathcal{H}_Q^-, j_Q^{-*})}(\mathcal{H}, e_Q(\sigma)) \rightarrow \text{Hom}_{(\mathcal{H}_{P_2 \cap Q}^-, j_{P_2 \cap Q}^{P_2 - *})}(\mathcal{H}_{P_2}^-, \sigma)$$

defined by $\varphi \mapsto \varphi \circ j_{P_2}^{-*}$. Then obviously this is $(\mathcal{H}_{P_2}^-, j_{P_2}^{-*})$ -equivariant.

Let $\lambda_{P_2}^-$ as in Proposition 2.5. Then we have $\mathcal{H}_{P_2}^-(T_{\lambda_{P_2}^-}^{P_2})^{-1} = \mathcal{H}_{P_2}$. By Lemma 3.16, we have $\mathcal{H}_{P_2 \cap Q}^-(T_{\lambda_{P_2}^-}^{P_2 \cap Q})^{-1} = \mathcal{H}_{P_2 \cap Q}^{P_2 -}$. Therefore we have

$$I_{P_2 \cap Q}^{P_2}(\sigma) = \text{Hom}_{(\mathcal{H}_{P_2 \cap Q}^{P_2 -}, j_{P_2 \cap Q}^{P_2 - *})}(\mathcal{H}_{P_2}, \sigma) = \text{Hom}_{(\mathcal{H}_{P_2 \cap Q}^-, j_{P_2 \cap Q}^{P_2 - *})}(\mathcal{H}_{P_2}^-, \sigma).$$

Hence we get an $(\mathcal{H}_{P_2}^-, j_{P_2}^{-*})$ -homomorphism $I_Q(e_Q(\sigma)) \rightarrow I_{P_2 \cap Q}^{P_2}(\sigma)$. For $v \in W_0^Q = W_{0, P_2}^{P_2 \cap Q}$, we have $\varphi(T_{n_v}) = (\varphi \circ j_{P_2}^{-*})(T_{n_v}^{P_2})$ by Corollary 2.7. Hence this homomorphism is an isomorphism by Proposition 2.9.

Let $w \in W_{P, \text{aff}}(1)$, $v \in W_0^Q$ and $\varphi \in I_Q(e_Q(\sigma))$. We have $W_0^Q \subset W_{0, P_2}$. Since $n_v \in W_{P_2, \text{aff}}(1)$, $\ell(w n_v) = \ell(w) + \ell(n_v)$. The subgroup $W_{P, \text{aff}}(1)$ is normal in $W(1)$ [AHHV, II.7 Remark 4]. Hence $n_v^{-1} w n_v \in W_{P, \text{aff}}(1)$ and $\ell(w n_v) = \ell(n_v) + \ell(n_v^{-1} w n_v)$. Therefore $(\varphi T_w^*)(T_{n_v}^*) = \varphi(T_w^* T_{n_v}^*) = \varphi(T_{w n_v}^*) = \varphi(T_{n_v}^* T_{n_v^{-1} w n_v}^*) = \varphi(T_{n_v}^*) e_Q(\sigma)(T_{n_v^{-1} w n_v}^{Q*})$. By the definition of the extension, $e_Q(\sigma)(T_{n_v^{-1} w n_v}^{Q*}) = 1$. Hence $(\varphi T_w^*)(T_{n_v}^*) = \varphi(T_{n_v}^*)$. Since an element in $I_Q(e_Q(\sigma))$ is determined by the values at $T_{n_v}^*$ for $v \in W_0^Q$ (Proposition 2.9), T_w^* acts trivially on $I_Q(e_Q(\sigma))$. Hence $I_Q(e_Q(\sigma)) \simeq e_G(I_{P_2 \cap Q}^{P_2}(\sigma))$. \square

By this lemma and Lemma 3.13, we have an \mathcal{H} -module structure on $I_Q(\mathbf{1}) \otimes e_G(\sigma)$.

Proposition 3.17. *We have $I_Q(e_Q(\sigma)) \simeq I_Q(\mathbf{1}) \otimes e_G(\sigma)$.*

Proof. Define the homomorphism $I_Q(\mathbf{1}) \otimes e_G(\sigma) \rightarrow e_Q(\sigma)$ by $\varphi \otimes x \mapsto \varphi(1)x$. For $w \in W_Q^-(1)$, we have

$$\begin{aligned} (\varphi \otimes x) T_w^* &= \varphi T_w^* \otimes x e_G(\sigma)(T_w^*) \mapsto (\varphi T_w^*)(1) x e_G(\sigma)(T_w^*) \\ &= \varphi(1) \mathbf{1}(T_w^{Q*}) x e_G(\sigma)(T_w^*) \\ &= \varphi(1) x e_G(\sigma)(T_w^*). \end{aligned}$$

By [Abe, Proposition 4.19], we have $e_G(\sigma)(T_w^*) = e_Q(\sigma)(T_w^{Q*})$. Hence the homomorphism is $(\mathcal{H}_Q^-, j_Q^{-*})$ -equivariant. Therefore we get a homomorphism $I_Q(\mathbf{1}) \otimes e_G(\sigma) \rightarrow I_Q(e_Q(\sigma))$. By Proposition 2.9, we have the decompositions $I_Q(\mathbf{1}) \otimes e_G(\sigma) = \bigoplus_{w \in W_0^Q} C \otimes e_G(\sigma)$ and $I_Q(e_Q(\sigma)) = \bigoplus_{w \in W_0^Q} e_Q(\sigma)$ which

makes the following diagram commutative

$$\begin{array}{ccc}
 I_Q(\mathbf{1}) \otimes e_G(\sigma) & \longrightarrow & I_Q(e_Q(\sigma)) \\
 \downarrow \wr & & \downarrow \wr \\
 \bigoplus_{w \in W_0^Q} e_G(\sigma) & \xlongequal{\quad} & \bigoplus_{w \in W_0^Q} e_G(\sigma)
 \end{array}$$

Hence we get the proposition. \square

4. INDUCTIONS

Let P be a parabolic subgroup. Recall that our parabolic induction I_P is defined by

$$I_P(\sigma) = \text{Hom}_{(\mathcal{H}_P^-, j_P^{-*})}(\mathcal{H}, \sigma)$$

for an \mathcal{H}_P -module σ . Since we also have two subalgebras \mathcal{H}_P^\pm and four homomorphisms $j_P^\pm, j_P^{\pm*}$, we can define four “inductions”. In this section, we study the relations between such functors.

4.1. Modules $\sigma_{\ell-\ell_P}$. Before studying such functors, we first consider the representation $\sigma_{\ell-\ell_P}$ attached to a representation σ of \mathcal{H}_P where P is a parabolic subgroup.

Let P be a parabolic subgroup and σ an \mathcal{H}_P -module. We define a linear map $\sigma_{\ell-\ell_P}$ by

$$\sigma_{\ell-\ell_P}(T_w^P) = (-1)^{\ell(w)-\ell_P(w)} \sigma(T_w^P)$$

for $w \in W_P(1)$. From the following lemma, this is again an \mathcal{H}_P -module.

Lemma 4.1. (1) *The linear map $\mathcal{H}_P \rightarrow C$ defined by $T_w^P \mapsto (-1)^{\ell(w)}$ is a character of \mathcal{H}_P and it sends T_s to -1 for any $s \in S_{\text{aff},P}(1)$.*
 (2) *Let π be an \mathcal{H} -module, χ a character of \mathcal{H} such that $\chi(T_t) = 1$ for any $t \in Z_\kappa$ and $\chi(T_s) = -1$ for any $s \in S_{\text{aff}}(1)$. Then the linear map defined by $T_w \mapsto (-1)^{\ell(w)} \chi(T_w) \pi(T_w)$ is an \mathcal{H} -module.*

Proof. (1) First we prove the last claim. Let $s \in S_{\text{aff},P}(1)$. Then s is a reflection in W . Hence $\ell(s)$ is odd. Therefore we have $(-1)^{\ell(s)} = -1$.

We check the quadratic relations. Let $s \in S_{\text{aff},P}(1)$. The quadratic relation is $(T_s^P)^2 = q_{s,P} T_{s^2}^P + c_s T_s^P$ and the left hand side goes to 1 by the map in the lemma. Since $s^2 \in Z_\kappa$, $T_{s^2}^P$ goes to 1. Hence $q_{s,P} T_{s^2}^P$ goes to $q_{s,P}$. The image of c_s under the map is $q_{s,P} - 1$ by Proposition 2.1. We have already proved that T_s^P goes to -1 . Hence the right hand side goes to $q_{s,P} + (q_{s,P} - 1)(-1) = 1$. Therefore the map preserves the quadratic relations.

We check the braid relations. Let $w_1, w_2 \in W_P(1)$. Then $T_{w_1 w_2}^P$ goes to $(-1)^{\ell(w_1 w_2)} = (-1)^{\ell(w_1) + \ell(w_2)}$ which is the product of the images of $T_{w_1}^P$ and $T_{w_2}^P$. Hence the map preserves the braid relations. We get (1).

(2) Let π' be the map given in (2) and first we check that π' preserves the quadratic relations. Let $s \in S_{\text{aff}}(1)$. We prove $\pi'(T_s)^2 = \pi'(q_s T_{s^2}) + \pi'(c_s T_s)$. We calculate the right hand side. Since $s^2 \in Z_\kappa$, we have $\chi(T_{s^2}) = 1$. We also have $\ell(s^2) = 0$. Hence $\pi'(T_{s^2}) = \pi(T_{s^2})$. Take $c_s(t) \in \mathbb{Z}$ such

that $c_s = \sum_{t \in Z_\kappa} c_s(t) T_t$. Then we have $\pi'(c_s T_s) = \sum_{t \in Z_\kappa} c_s(t) \pi'(T_t T_s) = \sum_{t \in Z_\kappa} c_s(t) (-1)^{\ell(ts)} \chi(T_t) \chi(T_s) \pi(T_t) \pi(T_s)$. Since $t \in Z_\kappa$, we have $\ell(ts) = \ell(s) = 1$. We also have $\chi(T_t) = 1$ by the assumption. Hence $\pi'(c_s T_s) = -\chi(T_s) \sum_{t \in Z_\kappa} c_s(t) \pi(T_t) \pi(T_s) = -\chi(T_s) \pi(c_s) \pi(T_s)$. Therefore $\pi'(q_s T_{s^2}) + \pi'(c_s T_s) = q_s \pi(T_{s^2}) - \chi(T_s) \pi(c_s) \pi(T_s)$. Since we assume $\chi(T_s) = -1$, we get $q_s \pi(T_{s^2}) - \chi(T_s) \pi(c_s) \pi(T_s) = q_s \pi(T_{s^2}) + \pi(c_s) \pi(T_s) = \pi(q_s T_{s^2} + c_s T_s) = \pi(T_s^2) = \pi(T_s)^2$. We have $\pi'(T_s)^2 = \chi(T_s)^2 \pi(T_s)^2$. By the assumption, $\chi(T_s) = -1$. Hence $\pi'(T_s)^2 = \pi(T_s)^2$. We get the quadratic relations.

Let $w_1, w_2 \in W_P(1)$ such that $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. Then

$$\begin{aligned} \pi'(T_{w_1} T_{w_2}) &= (-1)^{\ell(w_1 w_2)} \chi(T_{w_1 w_2}) \pi(T_{w_1 w_2}) \\ &= (-1)^{\ell(w_1) + \ell(w_2)} \chi(T_{w_1} T_{w_2}) \pi(T_{w_1} T_{w_2}) \\ &= (-1)^{\ell(w_1)} (-1)^{\ell(w_2)} \chi(T_{w_1}) \chi(T_{w_2}) \pi(T_{w_1}) \pi(T_{w_2}). \end{aligned}$$

Since $\chi(T_{w_2})$ is a scalar, we have

$$\begin{aligned} &(-1)^{\ell(w_1)} (-1)^{\ell(w_2)} \chi(T_{w_1}) \chi(T_{w_2}) \pi(T_{w_1}) \pi(T_{w_2}) \\ &= (-1)^{\ell(w_1)} \chi(T_{w_1}) \pi(T_{w_1}) (-1)^{\ell(w_2)} \chi(T_{w_2}) \pi(T_{w_2}) \\ &= \pi'(T_{w_1}) \pi'(T_{w_2}). \end{aligned}$$

Hence π' preserves the braid relations. \square

Let $\iota = \iota_G: \mathcal{H} \rightarrow \mathcal{H}$ be a linear map defined by $T_w \mapsto (-1)^{\ell(w)} T_w^*$ for $w \in W(1)$. Then this is an involution [Vig16, Proposition 4.23]. For any \mathcal{H} -module π , set $\pi^\iota = \pi \circ \iota$.

Lemma 4.2. *We have $(\sigma^{\iota_P})_{\ell-\ell_P} = (\sigma_{\ell-\ell_P})^{\iota_P}$.*

Proof. We prove $(\sigma^{\iota_P})_{\ell-\ell_P}(T_w^P) = (\sigma_{\ell-\ell_P})^{\iota_P}(T_w^P)$ for any $w \in W_P(1)$. We may assume that $w \in S_{\text{aff}, P}(1)$ or $\ell_P(w) = 0$.

First assume that $w \in S_{\text{aff}, P}(1)$ and denote w by s . By Lemma 4.1 (1), we have $(-1)^{\ell(s)} = -1$. Hence $(\sigma^{\iota_P})_{\ell-\ell_P}(T_s^P) = (-1)^{\ell(s)-\ell_P(s)} \sigma(\iota_P(T_s^P)) = \sigma(\iota_P(T_s^P)) = -\sigma(T_s^{P*})$. On the other hand, we have $(\sigma_{\ell-\ell_P})^{\iota_P}(T_s^P) = -(\sigma_{\ell-\ell_P})(T_s^{P*}) = -(\sigma_{\ell-\ell_P})(T_s^P) + (\sigma_{\ell-\ell_P})(c_s)$. We have $(\sigma_{\ell-\ell_P})(T_s^P) = (-1)^{\ell(s)-\ell_P(s)} \sigma(T_s^P) = \sigma(T_s^P)$ since $(-1)^{\ell(s)} = -1$ and $\ell_P(s) = 1$. We have $(\sigma_{\ell-\ell_P})(c_s) = \sigma(c_s)$. Hence $-(\sigma_{\ell-\ell_P})(T_s^P) + (\sigma_{\ell-\ell_P})(c_s) = -\sigma(T_s^P - c_s) = -\sigma(T_s^{P*})$.

Next assume that $\ell_P(w) = 0$. Then $\iota_P(T_w^P) = (-1)^{\ell_P(w)} T_w^{P*} = T_w^P$. Hence we have $(\sigma_{\ell-\ell_P})^{\iota_P} = \sigma_{\ell-\ell_P}(T_w^P) = (-1)^{\ell(w)-\ell_P(w)} \sigma(T_w^P)$. We also have $(\sigma^{\iota_P})_{\ell-\ell_P}(T_w^P) = (-1)^{\ell(w)-\ell_P(w)} \sigma(\iota_P(T_w^P)) = (-1)^{\ell(w)-\ell_P(w)} \sigma(T_w^P)$. We get the lemma. \square

By Lemma 4.2, we have $(\sigma^{\iota_P})_{\ell-\ell_P} = (\sigma_{\ell-\ell_P})^{\iota_P}$. We denote it by $\sigma_{\ell-\ell_P}^{\iota_P}$.

Lemma 4.3. *We have $(\sigma_{\ell-\ell_P})_{\ell-\ell_P} = \sigma$ and $(\sigma^{\iota_P})^{\iota_P} = \sigma$.*

Proof. Obvious from the definition and $\iota_P^2 = \text{id}$. \square

Lemma 4.4. *We have $(\sigma_{\ell-\ell_P}^{\iota_P})_{\ell-\ell_P}^{\iota_P} = \sigma$.*

Proof. The lemma follows from the above lemma and Lemma 4.2. \square

Lemma 4.5. *Let σ be an \mathcal{H}_P -module and $w \in W_P(1)$.*

- (1) We have $\sigma_{\ell-\ell_P}(T_w^{P*}) = (-1)^{\ell(w)-\ell_P(w)}\sigma(T_w^{P*})$.
 (2) For any orientation o , $\sigma_{\ell-\ell_P}(E_o^P(w)) = (-1)^{\ell(w)-\ell_P(w)}\sigma(E_o^P(w))$.

Proof. By Lemma 4.2, we have

$$\begin{aligned}\sigma_{\ell-\ell_P}(T_w^{P*}) &= (-1)^{\ell_P(w)}(\sigma_{\ell-\ell_P})^{\iota_P}(T_w^P) \\ &= (-1)^{\ell_P(w)}(\sigma^{\iota_P})_{\ell-\ell_P}(T_w^P) \\ &= (-1)^{\ell(w)-\ell_P(w)}(-1)^{\ell_P(w)}\sigma^{\iota_P}(T_w^P) \\ &= (-1)^{\ell(w)-\ell_P(w)}\sigma(T_w^{P*}).\end{aligned}$$

Now we prove (2) by induction on the length of w . If $\ell_P(w) = 0$, then $E_o^P(w) = T_w^P$. Hence the lemma follows from the definition of $\sigma_{\ell-\ell_P}$.

Assume that $\ell_P(w) > 0$ and take $s \in S_{\text{aff},P}(1)$ such that $\ell_P(s^{-1}w) < \ell_P(w)$. We have $E_o^P(w) = E_o^P(s)E_{o \cdot s}^P(s^{-1}w)$. Since $E_o^P(s)$ is T_s^P or T_s^{P*} , we have $\sigma_{\ell-\ell_P}(E_o^P(s)) = (-1)^{\ell(s)-\ell_P(s)}\sigma(E_o^P(s))$ as we have already proved. By inductive hypothesis,

$$\begin{aligned}\sigma_{\ell-\ell_P}(E_o^P(w)) &= \sigma_{\ell-\ell_P}(E_o^P(s))\sigma_{\ell-\ell_P}(E_{o \cdot s}^P(s^{-1}w)) \\ &= (-1)^{\ell(s)-\ell_P(s)}\sigma(E_o^P(s))(-1)^{\ell(s^{-1}w)-\ell_P(s^{-1}w)}\sigma(E_{o \cdot s}^P(s^{-1}w)).\end{aligned}$$

Since $(-1)^{\ell(s)-\ell_P(s)}(-1)^{\ell(s^{-1}w)-\ell_P(s^{-1}w)} = (-1)^{\ell(w)-\ell_P(w)}$, we have

$$\begin{aligned}\sigma_{\ell-\ell_P}(E_o^P(w)) &= (-1)^{\ell(w)-\ell_P(w)}\sigma(E_o^P(s))\sigma(E_{o \cdot s}^P(s^{-1}w)) \\ &= (-1)^{\ell(w)-\ell_P(w)}\sigma(E_o^P(s)E_{o \cdot s}^P(s^{-1}w)) \\ &= (-1)^{\ell(w)-\ell_P(w)}\sigma(E_o^P(w)).\end{aligned}$$

We get the lemma. \square

Remark 4.6. Applying Lemma 4.1 to the right regular representation of \mathcal{H}_P (namely, $\pi: \mathcal{H}_P \rightarrow \text{End}(\mathcal{H}_P)^{\text{op}}$ defined by $Y\pi(X) = YX$), we get the following: the linear map $\mathcal{H}_P \rightarrow \mathcal{H}_P$ defined by $T_w^P \mapsto (-1)^{\ell(w)-\ell_P(w)}T_w^P$ is an algebra homomorphism. By Lemma 4.5, this map sends T_w^{P*} and $E_o^P(w)$ to $(-1)^{\ell(w)-\ell_P(w)}T_w^{P*}$ and $(-1)^{\ell(w)-\ell_P(w)}E_o^P(w)$, respectively where o is any orientation.

Lemma 4.7. *The algebra homomorphism defined in the remark is identity on $\mathcal{H}_{\text{aff},P}$. Hence $\sigma_{\ell-\ell_P}|_{\mathcal{H}_{\text{aff},P}} = \sigma|_{\mathcal{H}_{\text{aff},P}}$ for any \mathcal{H}_P -module σ .*

Proof. The algebra $\mathcal{H}_{\text{aff},P}$ is generated by T_s where $s \in S_{\text{aff},P}(1)$. By Lemma 4.1, $(-1)^{\ell(s)} = -1$. Since the image of s in W_P is an affine simple reflection, we have $\ell_P(s) = 1$. Hence $(-1)^{\ell_P(s)} = -1$. \square

Lemma 4.8. *We have $\Delta(\sigma_{\ell-\ell_P}^{\iota_P}) = \Delta(\sigma)$.*

Proof. Let $\alpha \in \Delta(\sigma) \setminus \Delta_P$, P_α a parabolic subgroup corresponding to $\{\alpha\}$ and $\lambda \in \Lambda(1) \cap W_{\text{aff},P_\alpha}(1)$ and we prove $\sigma_{\ell-\ell_P}^{\iota_P}(T_\lambda^P) = 1$. Since $\ell_P(\lambda) = 0$ by Lemma 2.12, we have $\iota_P(T_\lambda^P) = (-1)^{\ell_P(\lambda)}T_\lambda^{P*} = T_\lambda^P$. Hence $\sigma_{\ell-\ell_P}^{\iota_P}(T_\lambda^P) = (-1)^{\ell(\lambda)-\ell_P(\lambda)}\sigma(T_\lambda^P)$. We have $\ell_P(\lambda) = 0$. Since $\lambda \in \Lambda(1) \cap W_{\text{aff}}(1)$, $\ell(\lambda)$ is even by Lemma 2.14. Hence $\sigma_{\ell-\ell_P}^{\iota_P}(T_\lambda^P) = \sigma(T_\lambda^P) = 1$. Therefore $\Delta(\sigma) \subset \Delta(\sigma_{\ell-\ell_P}^{\iota_P})$. Applying this to $\sigma_{\ell-\ell_P}^{\iota_P}$, by Lemma 4.4, we have $\Delta(\sigma_{\ell-\ell_P}^{\iota_P}) \subset \Delta(\sigma)$. \square

Lemma 4.9. *Let $P_1 \supset P$ be parabolic subgroups of G and σ an \mathcal{H}_P -module*

- (1) *We have $I_P^{P_1}(\sigma_{\ell_{P_1}-\ell_P})_{\ell-\ell_{P_1}} \simeq I_P^{P_1}(\sigma_{\ell-\ell_P})$.*
- (2) *For parabolic subgroups $Q \subset Q_1$ between P and $P(\sigma)$, we have $\text{St}_Q^{Q_1}(\sigma_{\ell_{Q_1}-\ell_P})_{\ell-\ell_{Q_1}} \simeq \text{St}_Q^{Q_1}(\sigma_{\ell-\ell_P})$.*

Proof. (1) Define a linear map $f: \mathcal{H}_{P_1} \rightarrow \mathcal{H}_{P_1}$ by $T_w^{P_1} \mapsto (-1)^{\ell(w)-\ell_{P_1}(w)} T_w^{P_1}$. Then by Remark 4.6, f is an algebra homomorphism. Put $\varphi^f = \varphi \circ f$ for $\varphi \in I_P^{P_1}(\sigma_{\ell_{P_1}-\ell_P})_{\ell-\ell_{P_1}}$. Then for $X \in \mathcal{H}_{P_1}$ and $w \in W_P^{P_1-}(1)$, we have

$$\begin{aligned}
\varphi^f(X j_P^{P_1-*}(T_w^{P*})) &= \varphi^f(X T_w^{P_1*}) \\
&= \varphi(f(X) f(T_w^{P_1*})) \\
&= (-1)^{\ell(w)-\ell_{P_1}(w)} \varphi(f(X) T_w^{P_1*}) \\
&= (-1)^{\ell(w)-\ell_{P_1}(w)} \varphi(f(X)) \sigma_{\ell_{P_1}-\ell_P}(T_w^{P*}) \\
&= (-1)^{\ell(w)-\ell_P(w)} \varphi(f(X)) \sigma(T_w^{P*}) \\
&= \varphi(f(X)) \sigma_{\ell-\ell_P}(T_w^{P*}) \\
&= \varphi^f(X) \sigma_{\ell-\ell_P}(T_w^{P*}).
\end{aligned}$$

Hence $\varphi^f \in I_P^{P_1}(\sigma_{\ell-\ell_P})$. For $w \in W_{P_1}(1)$ and $X \in \mathcal{H}_{P_1}$, we have

$$\begin{aligned}
(\varphi^f T_w^{P_1})(X) &= \varphi^f(T_w^{P_1} X) \\
&= \varphi(f(T_w^{P_1}) f(X)) \\
&= (-1)^{\ell(w)-\ell_{P_1}(w)} \varphi(T_w^{P_1} f(X)) \\
&= (\varphi T_w^{P_1})(f(X)) = (\varphi T_w^{P_1})^f(X).
\end{aligned}$$

Hence $\varphi \mapsto \varphi^f$ is an \mathcal{H} -module homomorphism, therefore an isomorphism $I_P^{P_1}(\sigma_{\ell_{P_1}-\ell_P})_{\ell-\ell_{P_1}} \rightarrow I_P^{P_1}(\sigma_{\ell-\ell_P})$.

(2) First we prove the lemma for $Q = Q_1$, namely $\text{St}_Q^{Q_1} = e_Q$. Let $w \in W_P(1)$. Then we have

$$\begin{aligned}
e_Q(\sigma_{\ell_Q-\ell_P})_{\ell-\ell_Q}(T_w^{Q*}) &= (-1)^{\ell(w)-\ell_Q(w)} e_Q(\sigma_{\ell_Q-\ell_P})(T_w^{Q*}) \\
&= (-1)^{\ell(w)-\ell_Q(w)} \sigma_{\ell_Q-\ell_P}(T_w^{P*}) \\
&= (-1)^{\ell(w)-\ell_Q(w)} (-1)^{\ell_Q(w)-\ell_P(w)} \sigma(T_w^{P*}) \\
&= (-1)^{\ell(w)-\ell_P(w)} \sigma(T_w^{P*}) = \sigma_{\ell-\ell_P}(T_w^{P*}).
\end{aligned}$$

If $w \in W_{Q \cap P_2, \text{aff}}(1)$ then $T_w^{Q*} \in \mathcal{H}_{\text{aff}, Q}$. Hence by Lemma 4.7, we have $e_Q(\sigma_{\ell_Q-\ell_P})_{\ell-\ell_Q}(T_w^{Q*}) = e_Q(\sigma_{\ell_Q-\ell_P})(T_w^{Q*})$. The definition of the extension says that it is 1. Therefore by a characterization of $e_Q(\sigma_{\ell-\ell_P})$, we have (2) in this case.

In general, consider the exact sequence

$$\bigoplus_{Q_1 \supset Q_2 \supset Q} I_{Q_2}^{Q_1}(e_{Q_2}(\sigma_{\ell_{Q_1}-\ell_P})) \rightarrow I_Q^{Q_1}(e_Q(\sigma_{\ell_{Q_1}-\ell_P})) \rightarrow \text{St}_Q^{Q_1}(\sigma_{\ell_{Q_1}-\ell_P}) \rightarrow 0.$$

Hence

$$\begin{aligned} \bigoplus_{Q_1 \supset Q_2 \supsetneq Q} I_{Q_2}^{Q_1}(e_{Q_2}(\sigma_{\ell_{Q_1}-\ell_P}))_{\ell-\ell_{Q_1}} &\rightarrow I_Q^{Q_1}(e_Q(\sigma_{\ell_{Q_1}-\ell_P}))_{\ell-\ell_{Q_1}} \\ &\rightarrow \text{St}_Q^{Q_1}(\sigma_{\ell_{Q_1}-\ell_P})_{\ell-\ell_{Q_1}} \rightarrow 0. \end{aligned}$$

Using (1) and (2) for $Q = Q_1$, for $Q_1 \supset Q_2 \supset Q$, we have

$$\begin{aligned} I_{Q_2}^{Q_1}(e_{Q_2}(\sigma_{\ell_{Q_1}-\ell_P}))_{\ell-\ell_{Q_1}} &= I_{Q_2}^{Q_1}(e_{Q_2}(\sigma_{\ell_{Q_2}-\ell_P})_{\ell_{Q_1}-\ell_{Q_2}})_{\ell-\ell_{Q_1}} \\ &= I_{Q_2}^{Q_1}(e_{Q_2}(\sigma_{\ell_{Q_2}-\ell_P}))_{\ell-\ell_{Q_2}} \\ &= I_{Q_2}^{Q_1}(e_{Q_2}(\sigma_{\ell-\ell_P})). \end{aligned}$$

Therefore

$$\bigoplus_{Q_1 \supset Q_2 \supsetneq Q} I_{Q_2}^{Q_1}(e_{Q_2}(\sigma_{\ell-\ell_P})) \rightarrow I_Q^{Q_1}(e_Q(\sigma_{\ell-\ell_P})) \rightarrow \text{St}_Q^{Q_1}(\sigma_{\ell_{Q_1}-\ell_P})_{\ell-\ell_{Q_1}} \rightarrow 0.$$

Hence we get the lemma. \square

4.2. The functor I'_P . We define the functor I'_P as follows.

Definition 4.10. For an \mathcal{H}_P -module σ , put

$$I'_P(\sigma) = \text{Hom}_{(\mathcal{H}_P^-, j_P^-)}(\mathcal{H}, \sigma).$$

We remark the following proposition.

Proposition 4.11. *Let P be a parabolic subgroup and σ an \mathcal{H}_P -module. Then the map $\varphi \mapsto \varphi \circ \iota$ induces an isomorphism $I_P(\sigma)^\iota \simeq I'_P(\sigma_{\ell-\ell_P}^{\iota_P})$.*

Proof. For $w \in W_P^-(1)$, we have

$$\begin{aligned} (\varphi \circ \iota)(X j_P^-(T_w^P)) &= (\varphi \circ \iota)(X T_w) \\ &= \varphi(\iota(X) \iota(T_w)) \\ &= (-1)^{\ell(w)} \varphi(\iota(X) T_w^*) \\ &= (-1)^{\ell(w)} \varphi(\iota(X)) \sigma(T_w^{P*}) \\ &= (-1)^{\ell(w)-\ell_P(w)} \varphi^\iota(X) \sigma^{\iota_P}(T_w^P). \end{aligned}$$

Hence $\varphi \circ \iota \in \text{Hom}_{(\mathcal{H}_P^-, j_P^-)}(\mathcal{H}, \sigma_{\ell-\ell_P}^{\iota_P})$. By the same argument implies that $\psi \mapsto \psi \circ \iota$ gives a homomorphism $I'_P(\sigma_{\ell-\ell_P}^{\iota_P}) \rightarrow I_P(\sigma)^\iota$ which is the inverse of the above homomorphism. \square

From the properties of I_P , we get the properties of I'_P .

Proposition 4.12. *We have the following.*

- (1) *The functor I'_P is exact.*
- (2) *The map $\varphi \mapsto (\varphi(T_{n_w}^*))$ gives an isomorphism $I'_P(\sigma) \simeq \bigoplus_{w \in W_0^P} \sigma$.*
- (3) *Let Q be a parabolic subgroup containing P . Then $I'_Q \circ I_P^{Q'} \simeq I'_P$ by the homomorphism $\varphi \mapsto (X \mapsto \varphi(X)(1))$.*

Proof. (1) follows from the exactness of I_P . Proposition 2.9 implies (2) and Proposition 2.10 implies (3). \square

4.3. Other inductions. The reason why we introduce only one induction I'_P is that the other two inductions are not new by the following proposition.

Proposition 4.13. *Put $P' = n_{w_G w_P} P^{\text{op}} n_{w_G w_P}^{-1}$.*

(1) *The map $\varphi \mapsto (X \mapsto \varphi(XT_{n_{w_G w_P}}))$ gives an isomorphism*

$$I'_P(\sigma) = \text{Hom}_{(\mathcal{H}_{P', j_{P'}^-}^-)}(\mathcal{H}, \sigma) \xrightarrow{\sim} \text{Hom}_{(\mathcal{H}_{P', j_{P'}^+}^+)}(\mathcal{H}, n_{w_G w_P} \sigma).$$

(2) *The map $\varphi \mapsto (X \mapsto \varphi(XT_{n_{w_G w_P}}^*))$ gives an isomorphism*

$$I_P(\sigma) = \text{Hom}_{(\mathcal{H}_{P, j_P^-}^-)}(\mathcal{H}, \sigma) \xrightarrow{\sim} \text{Hom}_{(\mathcal{H}_{P', j_{P'}^+}^+)}(\mathcal{H}, n_{w_G w_P} \sigma).$$

Remark 4.14. By $\varphi \mapsto \varphi \circ \iota$, we have

$$\text{Hom}_{(\mathcal{H}_{P, j_P^+}^+)}(\mathcal{H}, \sigma)^\iota \simeq \text{Hom}_{(\mathcal{H}_{P', j_{P'}^+}^+)}(\mathcal{H}, \sigma_{\ell-\ell_P}^{\iota_P}).$$

Hence the statement (2) follows from (1).

First we check that the map is a homomorphism. For the calculation, we need the following lemma. Recall the notation ${}^P W_0$ from subsection 2.9.

Lemma 4.15. *Let $w \in {}^P W_0, v \in W_{0,P}$ and $\lambda \in Z(W_P(1))Z_\kappa$. For $o = o_+$ or o_- , we have $E_{o \cdot v}(\lambda n_w) = E_o(\lambda n_w)$.*

Proof. We may assume $\lambda \in Z(W_P(1))$. We prove the lemma in $\mathcal{H}[q_s^{\pm 1}]$. First we assume $o = o_-$ and prove the lemma by induction on $\ell(w)$. Assume that $w = 1$. Take anti-dominant $\lambda_1, \lambda_2 \in Z(W_P(1))$ such that $\lambda = \lambda_1 \lambda_2^{-1}$. Then by (2.1), we have $E_{o_-}(\lambda) E_{o_-}(\lambda_2) = q_\lambda^{1/2} q_{\lambda_2}^{1/2} q_{\lambda_1}^{-1/2} E_{o_-}(\lambda_1)$. Since λ_1, λ_2 are anti-dominant, we have $E_{o_-}(\lambda_1) = T_{\lambda_1}$ and $E_{o_-}(\lambda_2) = T_{\lambda_2}$ by (2.3). Hence $E_{o_-}(\lambda) = q_\lambda^{1/2} q_{\lambda_1}^{-1/2} q_{\lambda_2}^{1/2} T_{\lambda_1} T_{\lambda_2}^{-1}$. Since $n_v^{-1} \cdot \lambda_1 = \lambda_1$ and $n_v^{-1} \cdot \lambda_2 = \lambda_2$ are both anti-dominant, we have $E_{o_- \cdot v}(\lambda) = q_\lambda^{1/2} q_{\lambda_1}^{-1/2} q_{\lambda_2}^{1/2} T_{\lambda_1} T_{\lambda_2}^{-1}$ by the same argument. Hence we get the lemma in this case.

Assume that $\ell(w) > 0$ and take $s \in S_0$ such that $ws < w$. Then by [Deo77, Lemma 3.1], we have $ws \in {}^P W_0$. By the product formula (2.1), we have

$$\begin{aligned} E_{o_-}(\lambda n_w) &= q_{\lambda n_w}^{1/2} q_{\lambda n_{ws}}^{-1/2} q_{n_s}^{-1/2} E_{o_-}(\lambda n_{ws}) E_{o_- \cdot ws}(n_s), \\ E_{o_- \cdot v}(\lambda n_w) &= q_{\lambda n_w}^{1/2} q_{\lambda n_{ws}}^{-1/2} q_{n_s}^{-1/2} E_{o_- \cdot v}(\lambda n_{ws}) E_{o_- \cdot vws}(n_s). \end{aligned}$$

By inductive hypothesis, we have $E_{o_-}(\lambda n_{ws}) = E_{o_- \cdot v}(\lambda n_{ws})$. Hence it is sufficient to prove that $E_{o_- \cdot ws}(n_s) = E_{o_- \cdot vws}(n_s)$. Since $ws < w$, we have $E_{o_- \cdot ws}(n_s) = T_{n_s}$ by (2.4). Take $\alpha \in \Delta$ such that $s = s_\alpha$. Then $ws(\alpha) = -w(\alpha) > 0$. If $vws(\alpha) < 0$, then $ws(\alpha) \in \Sigma_P^+$ since $v \in W_{0,P}$. Since $w \in {}^P W_0$, we have $-\alpha = w^{-1}(ws(\alpha)) \in w^{-1}(\Sigma_P^+) \subset \Sigma^+$. This is a contradiction. Hence $vws(\alpha) > 0$. Therefore $vws < vw$. By (2.4), we have $E_{o_- \cdot vws}(n_s) = T_{n_s}$. We get $E_{o_- \cdot ws}(n_s) = E_{o_- \cdot vws}(n_s)$ and finish the inductive step.

Now we get $E_{o_- \cdot v}(\lambda n_w) = E_{o_-}(\lambda n_w)$. Applying ι to both sides with [Vig16, Lemma 5.31], we get $(-1)^{\ell(\lambda n_w)} E_{o_+ \cdot v}(\lambda n_w) = (-1)^{\ell(\lambda n_w)} E_{o_+}(\lambda n_w)$. Hence $E_{o_+ \cdot v}(\lambda n_w) = E_{o_+}(\lambda n_w)$. \square

We start to prove Proposition 4.13.

Lemma 4.16. *The map given in Proposition 4.13 is an \mathcal{H} -module homomorphism.*

Proof. Put $n = n_{w_G w_P}$. The lemma is equivalent to that the map $\varphi \mapsto \varphi(T_n)$ from $I'_P(\sigma)$ to $n\sigma$ gives an $(\mathcal{H}_{P'}^+, j_{P'}^+)$ -module homomorphism. Let $w \in W_{P'}(1)$ be a P' -positive element. We have

$$(\varphi j_{P'}^+(E_{o_+, P'}^{P'}(w)))(T_n) = \varphi(j_{P'}^+(E_{o_+, P'}^{P'}(w))T_n).$$

and, by Lemma 2.6, we have

$$j_{P'}^+(E_{o_+, P'}^{P'}(w)) = j_{P'}^+(E_{o_-, P'}^{P'}(w)) = E_{o_- \cdot w_{P'}}(w).$$

We have $w_G w_P = w_{P'} w_G \in {}^P W_0$. Hence by Lemma 4.15, we have $T_n = E_{o_-}(n) = E_{o_- \cdot n_{w_{P'} w}}(n)$. By Lemma 2.18, we have $\ell(w) + \ell(n) = \ell(wn) = \ell(nn^{-1}wn) = \ell(n) + \ell(n^{-1}wn)$. Here we use that $n \in W_0^P$ and $n^{-1}wn \in W_P(1)$ is P -negative. Therefore

$$\begin{aligned} E_{o_- \cdot w_{P'}}(w)T_n &= E_{o_- \cdot w_{P'}}(w)E_{o_- \cdot n_{w_{P'} w}}(n) \\ &= E_{o_- \cdot w_{P'}}(wn) \\ &= E_{o_- \cdot w_{P'}}(n)E_{o_- \cdot n_{w_{P'} n}}(n^{-1}wn) \\ &= T_n E_{o_- \cdot n_{w_{P'} n}}(n^{-1}wn). \end{aligned}$$

Since $n_{w_{P'}} n = n_{w_{P'}} n_{w_G w_P} = n_{w_G}$, we have $o_- \cdot n_{w_{P'}} n = o_- \cdot w_G = o_+$. Hence $E_{o_- \cdot n_{w_{P'} n}}(n^{-1}wn) = E_{o_+}(n^{-1}wn) = j_P^-(E_{o_+, P}^P(n^{-1}wn))$ by Lemma 2.6. Therefore we have

$$(4.1) \quad j_{P'}^+(E_{o_+, P'}^{P'}(w))T_n = T_n j_P^-(E_{o_+, P}^P(n^{-1}wn)).$$

Hence we get

$$\begin{aligned} \varphi(j_{P'}^+(E_{o_+, P'}^{P'}(w))T_n) &= \varphi(T_n j_P^-(E_{o_+, P}^P(n^{-1}wn))) \\ &= \varphi(T_n)\sigma(E_{o_+, P}^P(n^{-1}wn)) \\ &= \varphi(T_n)(n\sigma)(E_{o_+, P'}^{P'}(w)). \end{aligned}$$

We get the lemma. \square

We construct the homomorphism in the opposite direction.

Lemma 4.17. *Let $\lambda = \lambda_P^- \in Z(W_P(1))$ as in Proposition 2.5. Put $n = n_{w_G w_P}$ and $P' = nP^{\text{op}}n^{-1}$. Then $\varphi \mapsto (X \mapsto \varphi(XE_{o_+}(\lambda n^{-1})))$ gives a homomorphism*

$$\text{Hom}_{(\mathcal{H}_{P'}^+, j_{P'}^+)}(\mathcal{H}, n\sigma) \rightarrow \text{Hom}_{(\mathcal{H}_P^-, j_P^-)}(\mathcal{H}, \sigma) = I'_P(\sigma).$$

Proof. We prove that $\varphi \mapsto \varphi(E_{o_+}(\lambda n^{-1}))$ is an (\mathcal{H}_P^-, j_P^-) -homomorphism $\text{Hom}_{(\mathcal{H}_{P'}^+, j_{P'}^+)}(\mathcal{H}, n\sigma) \rightarrow \sigma$. Let $w \in W_P(1)$ be a P -negative element. Then

$$(\varphi j_P^-(E_{o_+, P}^P(w)))(E_{o_+}(\lambda n^{-1})) = \varphi(j_P^-(E_{o_+, P}^P(w))E_{o_+}(\lambda n^{-1}))$$

and $j_P^-(E_{o_+, P}^P(w)) = E_{o_+}(w)$ by Lemma 2.6. Since $w \in W_P(1)$, $\lambda \in Z(W_P(1))$ and $(w_G w_P)^{-1} \in {}^P W_0$, we have $E_{o_+}(\lambda n^{-1}) = E_{o_+ \cdot w}(\lambda n^{-1})$ by Lemma 4.15. We also have, by Lemma 2.19, $\ell(w) + \ell(\lambda n^{-1}) = \ell(w\lambda n^{-1})$. Hence

$$E_{o_+}(w)E_{o_+}(\lambda n^{-1}) = E_{o_+}(w)E_{o_+ \cdot w}(\lambda n^{-1}) = E_{o_+}(w\lambda n^{-1}).$$

Since $\lambda \in Z(W_P(1))$, we have $w\lambda n^{-1} = \lambda w n^{-1} = \lambda n^{-1}(nwn^{-1})$. The element nwn^{-1} is P' -positive. By Lemma 2.19, $\ell(\lambda n^{-1}nwn^{-1}) = \ell(\lambda n^{-1}) + \ell(nwn^{-1})$. (We have $\lambda n^{-1} = n^{-1}(n\lambda n^{-1})$ and since $n(\Sigma^+ \setminus \Sigma_P^+) = \Sigma^- \setminus \Sigma_{P'}^-$, $n\lambda n^{-1}$ satisfies the condition of $\lambda_{P'}^+$ in Proposition 2.5. We also have $(w_G w_P)^{-1} = w_P w_G = w_G w_{P'} \in W_0^{P'}$.) Hence we have $E_{o_+}(w\lambda n^{-1}) = E_{o_+}(\lambda n^{-1})E_{o_+ \cdot n^{-1}}(nwn^{-1})$. We have $o_+ \cdot n^{-1} = o_+ \cdot w_P w_G = o_+ \cdot w_G w_{P'} = o_- \cdot w_{P'}$. Hence we have

$$\begin{aligned} E_{o_+ \cdot n^{-1}}(nwn^{-1}) &= E_{o_- \cdot w_{P'}}(nwn^{-1}) \\ &= j_{P'}^+(E_{o_-, P' \cdot w_{P'}}^{P'}(nwn^{-1})) \\ &= j_{P'}^+(E_{o_+, P'}^{P'}(nwn^{-1})) \end{aligned}$$

by Lemma 2.6. Therefore, we have

$$(4.2) \quad j_P^-(E_{o_+, P}^P(w))E_{o_+}(\lambda n^{-1}) = E_{o_+}(\lambda n^{-1})j_{P'}^+(E_{o_+, P'}^{P'}(nwn^{-1}))$$

Therefore

$$\begin{aligned} (\varphi j_P^-(E_{o_+, P}^P(w)))(E_{o_+}(\lambda n^{-1})) &= \varphi(j_P^-(E_{o_+, P}^P(w))(E_{o_+}(\lambda n^{-1}))) \\ &= \varphi(E_{o_+}(\lambda n^{-1})j_{P'}^+(E_{o_+, P'}^{P'}(nwn^{-1}))) \\ &= \varphi(E_{o_+}(\lambda n^{-1}))(n\sigma)(E_{o_+, P'}^{P'}(nwn^{-1})) \\ &= \varphi(E_{o_+}(\lambda n^{-1}))\sigma(E_{o_+, P}^P(w)). \end{aligned}$$

We get the lemma. \square

Proof of Proposition 4.13. We prove that the compositions of the homomorphisms in Proposition 4.13 and Lemma 4.17 are isomorphisms. Let Φ be the homomorphism in Proposition 4.13 and Ψ that in Lemma 4.17.

Put $n = n_{w_G w_P}$ and $P' = nP^{\text{op}}n^{-1}$. For $\varphi \in \text{Hom}_{(\mathcal{H}_{P'}, j_{P'}^+)}(\mathcal{H}, n\sigma)$, $\Phi(\Psi(\varphi))$ is given by

$$\Phi(\Psi(\varphi))(X) = \Psi(\varphi)(XT_n) = \varphi(XT_n E_{o_+}(\lambda n^{-1}))$$

where $\lambda = \lambda_P^-$ as in Proposition 2.5. We have $T_n = E_{o_-}(n)$ by (2.4). Since $w_G w_P = w_{P'} w_G \in {}^{P'}W_0$, we have $E_{o_-}(n) = E_{o_- \cdot w_{P'}}(n)$ by Lemma 4.15. We have $o_- \cdot w_{P'} n = o_- \cdot w_G = o_+$. Since $(w_G w_P)^{-1} = w_P w_G \in {}^P W_0$, by Lemma 2.19 and 2.15, we have $\ell(\lambda n^{-1}) = \ell(\lambda) - \ell(n) = \ell(n \cdot \lambda) - \ell(n)$. Hence, by Lemma 2.6, we have

$$\begin{aligned} T_n E_{o_+}(\lambda n^{-1}) &= E_{o_- \cdot w_{P'}}(n)E_{o_- \cdot w_{P'} n}(\lambda n^{-1}) \\ &= E_{o_- \cdot w_{P'}}(n\lambda n^{-1}) \\ (4.3) \quad &= j_{P'}^+(E_{o_-, P' \cdot w_{P'}}^{P'}(n\lambda n^{-1})) \\ &= j_{P'}^+(E_{o_+, P'}^{P'}(n\lambda n^{-1})). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \Phi(\Psi(\varphi))(X) &= \varphi(Xj_{P'}^+(E_{o_+, P'}^{P'}(n\lambda n^{-1}))) \\ &= \varphi(X)(n\sigma)(E_{o_+, P'}^{P'}(n\lambda n^{-1})) \\ &= \varphi(X)\sigma(E_{o_+, P}^P(\lambda)). \end{aligned}$$

Since λ is in $Z(W_P(1))$, $\sigma(E_{o_+,P}^P(\lambda))$ is invertible. Hence $\Phi \circ \Psi$ is invertible.

Next, for $\psi \in I_P'(\sigma) = \text{Hom}_{(\mathcal{H}_P^-, j_P^-)}(\mathcal{H}, \sigma)$, we have

$$\Psi(\Phi(\psi))(X) = \psi(X E_{o_+}(\lambda n^{-1}) T_n).$$

As in the above argument, we have $T_n = E_{o_-, w_{P'}}(n) = E_{o_+, n^{-1}}(n)$. We also have $\ell(\lambda) = \ell(\lambda n^{-1}) + \ell(n)$ as in the above. We have

$$(4.4) \quad E_{o_+}(\lambda n^{-1}) T_n = E_{o_+}(\lambda n^{-1}) E_{o_+, n^{-1}}(n) = E_{o_+}(\lambda) = j_P^-(E_{o_+,P}^P(\lambda)).$$

Therefore we have

$$\Psi(\Phi(\psi))(X) = \psi(X j_P^-(E_{o_+,P}^P(\lambda))) = \psi(X) \sigma(E_{o_+,P}^P(\lambda)).$$

Since λ is in $Z(W_P(1))$, $\sigma(E_{o_+,P}^P(\lambda))$ is invertible. Hence $\Psi \circ \Phi$ is invertible. \square

4.4. Tensor products. Recall that we have

$$I_P(\sigma) \simeq n_{w_G w_P} \sigma \otimes_{(\mathcal{H}_{P'}^+, j_{P'}^+)} \mathcal{H}.$$

where $P' = n_{w_G w_P} P^{\text{op}} n_{w_G w_P}^{-1}$ by Proposition 2.21. Again, we can consider the four inductions defined via the tensor product. By $x \otimes X \mapsto x \otimes \iota(X)$, we have

$$(\sigma \otimes_{(\mathcal{H}_P^+, j_P^+)} \mathcal{H})^\iota \simeq \sigma_{\ell-P}^{\iota_P} \otimes_{(\mathcal{H}_P^+, j_P^{+*})} \mathcal{H}$$

and

$$(\sigma \otimes_{(\mathcal{H}_P^-, j_P^-)} \mathcal{H})^\iota \simeq \sigma_{\ell-P}^{\iota_P} \otimes_{(\mathcal{H}_P^-, j_P^{-*})} \mathcal{H}.$$

Proposition 4.18. *Let P be a parabolic subgroup and σ an \mathcal{H}_P -module. Put $P' = n_{w_G w_P} P^{\text{op}} n_{w_G w_P}^{-1}$.*

(1) *The map $x \otimes X \mapsto x \otimes T_{n_{w_G w_P}} X$ gives an isomorphism*

$$\sigma \otimes_{(\mathcal{H}_P^-, j_P^-)} \mathcal{H} \rightarrow n_{w_G w_P} \sigma \otimes_{(\mathcal{H}_{P'}^+, j_{P'}^+)} \mathcal{H}.$$

(2) *The map $x \otimes X \mapsto x \otimes T_{n_{w_G w_P}}^* X$ gives an isomorphism*

$$\sigma \otimes_{(\mathcal{H}_P^-, j_P^{-*})} \mathcal{H} \rightarrow n_{w_G w_P} \sigma \otimes_{(\mathcal{H}_{P'}^+, j_{P'}^{+*})} \mathcal{H}.$$

Proof. (2) follows from (1). We prove (1).

Put $n = n_{w_G w_P}$. Let Φ be a homomorphism given in the proposition and first we prove that this is a well-defined \mathcal{H} -homomorphism. We prove that the linear map $x \mapsto x \otimes T_n$ is an (\mathcal{H}_P^-, j_P^-) -homomorphism $\sigma \rightarrow n \sigma \otimes_{(\mathcal{H}_{P'}^+, j_{P'}^+)} \mathcal{H}$. Let $x \in \sigma$ and $w \in W_P^-(1)$. Then $nwn^{-1} \in W_{P'}^+(1)$. Hence by (4.1), we have

$$j_{P'}^+(E_{o_+,P'}^{P'}(nwn^{-1})) T_n = T_n j_P^-(E_{o_+,P}^P(w))$$

Hence, in $n \sigma \otimes_{(\mathcal{H}_{P'}^+, j_{P'}^+)} \mathcal{H}$, we have

$$\begin{aligned} x \otimes T_n j_P^-(E_{o_+,P}^P(w)) &= x \otimes j_{P'}^+(E_{o_+,P'}^{P'}(nwn^{-1})) T_n \\ &= x(n \sigma)(E_{o_+,P'}^{P'}(nwn^{-1})) \otimes T_n \\ &= x \sigma(E_{o_+,P}^P(w)) \otimes T_n. \end{aligned}$$

Therefore Φ is an \mathcal{H} -module homomorphism.

Next let $\lambda = \lambda_{\bar{P}}$ and consider the linear map $\Psi: x \otimes X \mapsto x \otimes E_{o_+}(\lambda n^{-1})X$. We prove that Ψ is also a well-defined \mathcal{H} -homomorphism $n\sigma \otimes_{(\mathcal{H}_{P'}^+, j_{P'}^+)} \mathcal{H} \rightarrow \sigma \otimes_{(\mathcal{H}_{\bar{P}}^-, j_{\bar{P}}^-)} \mathcal{H}$. Let $w \in W_{P'}^+(1)$ and $x \in \sigma$. Then $n^{-1}wn \in W_{\bar{P}}^-(1)$. By (4.2), we have

$$j_{\bar{P}}^-(E_{o_+,P}^P(n^{-1}wn))E_{o_+}(\lambda n^{-1}) = E_{o_+}(\lambda n^{-1})j_{P'}^+(E_{o_+,P'}^{P'}(w)).$$

Hence, in $\sigma \otimes_{(\mathcal{H}_{\bar{P}}^-, j_{\bar{P}}^-)} \mathcal{H}$, we have

$$\begin{aligned} x \otimes E_{o_+}(\lambda n^{-1})j_{P'}^+(E_{o_+,P'}^{P'}(w)) &= x \otimes j_{\bar{P}}^-(E_{o_+,P}^P(n^{-1}wn))E_{o_+}(\lambda n^{-1}) \\ &= x\sigma(E_{o_+,P}^P(n^{-1}wn)) \otimes E_{o_+}(\lambda n^{-1}) \\ &= x(n\sigma)(E_{o_+,P'}^{P'}(w)) \otimes E_{o_+}(\lambda n^{-1}). \end{aligned}$$

Therefore Ψ is an \mathcal{H} -homomorphism.

Let $x \in \sigma$ and $X \in \mathcal{H}$. By (4.3), we have

$$\begin{aligned} \Phi(\Psi(x \otimes X)) &= x \otimes T_n E_{o_+}(\lambda n^{-1})X \\ &= x \otimes j_{P'}^+(E_{o_+,P'}^{P'}(n\lambda n^{-1}))X \\ &= x(n\sigma)(E_{o_+,P'}^{P'}(n\lambda n^{-1})) \otimes X \\ &= x\sigma(E_{o_+,P}^P(\lambda)) \otimes X. \end{aligned}$$

Since $\lambda \in Z(W_P(1))$, $\sigma(E_{o_+,P}^P(\lambda))$ is invertible. Hence $\Phi \circ \Psi$ is invertible. By (4.4), we also have

$$\begin{aligned} \Psi(\Phi(x \otimes X)) &= x \otimes E_{o_+}(\lambda n^{-1})T_n X \\ &= x \otimes j_{\bar{P}}^-(E_{o_+,P}^P(\lambda))X \\ &= x\sigma(E_{o_+,P}^P(\lambda)) \otimes X. \end{aligned}$$

This is again invertible. \square

Corollary 4.19. *Let P be a parabolic subgroup and σ an \mathcal{H}_P -module. Then we have*

$$\begin{aligned} I_P(\sigma) &\simeq \sigma \otimes_{(\mathcal{H}_{\bar{P}}^-, j_{\bar{P}}^-)} \mathcal{H}, \\ I'_P(\sigma) &\simeq \sigma \otimes_{(\mathcal{H}_{\bar{P}}^-, j_{\bar{P}}^{-*})} \mathcal{H}. \end{aligned}$$

Proof. The first one follows from the Propositions 2.21 and 4.18 and the second one follows from the twist of the first one. \square

Proposition 4.20. *Let P be a parabolic subgroup and σ an \mathcal{H}_P -module.*

- (1) *The isomorphism $\sigma \otimes_{(\mathcal{H}_{\bar{P}}^-, j_{\bar{P}}^-)} \mathcal{H} \rightarrow I_P(\sigma)$ is given by the following. For $x \in \sigma$, let $\varphi_x \in I_P(\sigma)$ be an element such that $\varphi_x(1) = x$ and $\varphi_x(T_{n_w}^*) = 0$ for any $w \in W_0^P \setminus \{1\}$. Then the isomorphism is given by $x \otimes X \mapsto \varphi_x X$.*
- (2) *The isomorphism $\sigma \otimes_{(\mathcal{H}_{\bar{P}}^-, j_{\bar{P}}^{-*})} \mathcal{H} \rightarrow I'_P(\sigma)$ is given by the following. For $x \in \sigma$, let $\varphi_x \in I_P(\sigma)$ be an element such that $\varphi_x(1) = x$ and $\varphi_x(T_{n_w}) = 0$ for any $w \in W_0^P \setminus \{1\}$. Then the isomorphism is given by $x \otimes X \mapsto \varphi_x X$.*

In particular, these isomorphisms do not depend on a choice of a lift $n_{w_G w_P}$.

Proof. The second statement follows from the first one. From the construction, the image of $x \in \sigma$ under the isomorphism in Corollary 4.19 is given by $\psi_x T_{n_{w_G w_P}}$ where ψ_x is characterized by $\psi_x(T_{n_{w_G w_P}}) = x$ and $\psi_x(T_{n_w}) = 0$ for any $w \in W_0^P \setminus \{w_G w_P\}$. We prove $\psi_x T_{n_{w_G w_P}} = \varphi_x$.

Set $\psi = \psi_x T_{n_{w_G w_P}}$. We have $\psi(1) = \psi_x(T_{n_{w_G w_P}}) = x$. If $w \in W_0^P \setminus \{1\}$, then

$$\begin{aligned} T_{n_{w_G w_P}} T_{n_w}^* &= E_{o_+ \cdot (w_G w_P)^{-1}}(n_{w_G w_P}) E_{o_+}(n_w) \\ &= q_{w_G w_P}^{1/2} q_w^{1/2} q_{w_G w_P}^{-1/2} E_{o_+ \cdot (w_G w_P)^{-1}}(n_{w_G w_P} n_w) \\ &\in \sum_{v \in W_0, v \leq w_G w_P w} C[Z_\kappa] T_{n_v}. \end{aligned}$$

Hence it is sufficient to prove that if $v \leq w_G w_P w$ then $\psi_x(T_{n_v}) = 0$. Since $w \notin W_{P,0}$, we have $w_G w_P w \notin w_G w_P W_{P,0}$. Hence by [Abe, Lemma 4.13 (3)], we have $v \notin w_G w_P W_{P,0}$. Take $v_1 \in W_0^P$ and $v_2 \in W_{P,0}$ such that $v = v_1 v_2$. Then we have $\psi_x(T_{n_v}) = \psi_x(T_{n_{v_1}} T_{n_{v_2}}) = \psi_x(T_{n_{v_1}}) \sigma(T_{n_{v_2}}^P)$. Since $v_1 \neq w_G w_P$, this is zero. \square

5. ADJOINT FUNCTORS

5.1. Adjoint functors L_P and R_P . Let P be a parabolic subgroup. By the definition of the parabolic induction I_P , it has the left adjoint functor L_P . The functor L_P is defined by

$$L_P(\pi) = \pi \otimes_{(\mathcal{H}_{P^-}, j_{P^-}^*)} \mathcal{H}_P = \pi E_{o_-}(\lambda_P^-)^{-1}$$

where λ_P^- is as in Proposition 2.5. Since this is a localization, this functor is exact.

By Proposition 2.21, the functor I_P also has the right adjoint functor R_P . It is defined as follows. Set $P' = n_{w_G w_P} P^{\text{op}} n_{w_G w_P}^{-1}$. Then we have

$$R_P(\pi) = n_{w_G w_P}^{-1} \text{Hom}_{(\mathcal{H}_{P'}^+, j_{P'}^+)}(\mathcal{H}_{P'}, \pi).$$

This is left exact. Let $\lambda_{P'}^+$ be as in Proposition 2.5. By Proposition 2.5, we have $\mathcal{H}_{P'} = \mathcal{H}_{P'}^+(T_{\lambda_{P'}^+}^{P'})^{-1}$. Hence $\varphi \mapsto (\varphi((T_{\lambda_{P'}^+}^{P'})^{-n}))$ gives an isomorphism

$$R_P(\pi) \simeq \{(x_n)_{n \in \mathbb{Z}_{\geq 1}} \mid x_n \in \pi, x_{n+1} T_{\lambda_{P'}^+} = x_n\}.$$

Let P_1 be a parabolic subgroup containing P . Then the left adjoint functor (resp. the right adjoint functor) of $I_P^{P_1}$ is denoted by $L_P^{P_1}$ (resp. $R_P^{P_1}$).

5.2. Parabolic inductions and adjoint functors. In this subsection, we prove the following proposition. The condition on σ is found in the study in [AHVa].

Proposition 5.1. *Let P, Q be parabolic subgroups and σ an \mathcal{H}_Q -module. Assume that $\bigcap_{n \in \mathbb{Z}_{\geq 0}} p^n \sigma = 0$. Then we have $R_P \circ I_Q(\sigma) \simeq I_{P \cap Q}^P \circ R_{P \cap Q}^Q(\sigma)$.*

Before proving the proposition, we reformulate the proposition in terms of

$$\begin{aligned}\tilde{I}_Q(\sigma) &= \sigma \otimes_{(\mathcal{H}_Q^+, j_Q^+)} \mathcal{H}, \\ \tilde{R}_P(\pi) &= \text{Hom}_{(\mathcal{H}_P^+, j_P^+)}(\mathcal{H}_P, \pi).\end{aligned}$$

By Proposition 2.21, we have

$$I_Q(\sigma) = \tilde{I}_{Q'}(n_{w_G w_Q} \sigma), \quad R_P(\pi) = n_{w_G w_P}^{-1} \tilde{R}_{P'}(\pi)$$

where $P' = n_{w_G w_P} P^{\text{op}} n_{w_G w_P}^{-1}$ and $Q' = n_{w_G w_Q} Q^{\text{op}} n_{w_G w_Q}^{-1}$.

Lemma 5.2. *Let P, Q be a parabolic subgroup and σ an \mathcal{H}_Q -module. Assume that $\bigcap_{n \in \mathbb{Z}_{\geq 0}} p^n \sigma = 0$. Then we have $\tilde{R}_P \circ \tilde{I}_Q(\sigma) \simeq \tilde{I}_{P \cap Q}^P \circ \tilde{R}_{P \cap Q}^Q(\sigma)$.*

We prove that Lemma 5.2 implies Proposition 5.1. We have

$$R_P \circ I_Q = n_{w_G w_P}^{-1} \tilde{R}_{P'} \circ \tilde{I}_{Q'} n_{w_G w_Q}.$$

By Lemma 5.2, we have $\tilde{R}_{P'} \circ \tilde{I}_{Q'} = \tilde{I}_{P' \cap Q'}^{P'} \circ \tilde{R}_{P' \cap Q'}^{Q'}$. Let P_1 be a parabolic subgroup such that $\Delta_{P_1} = w_P(-\Delta_{P \cap Q}) = w_P w_{P \cap Q}(\Delta_{P \cap Q})$. Then we have $w_G w_P(\Delta_{P_1}) = w_G(-\Delta_{P \cap Q}) = w_G(-\Delta_P) \cap w_G(-\Delta_Q) = \Delta_{P'} \cap \Delta_{Q'} = \Delta_{P' \cap Q'}$. The adjoint action of $n_{w_G w_P}$ induces an isomorphism $\mathcal{H}_P \simeq \mathcal{H}_{P'}$ and it induces an isomorphism $\mathcal{H}_{P_1}^{P+}$ to $\mathcal{H}_{P' \cap Q'}^{P'+}$. Hence we get $n_{w_G w_P}^{-1} \tilde{R}_{P' \cap Q'}^{P'} = \tilde{R}_{P_1}^P n_{w_G w_P}^{-1}$. Similarly we have $\tilde{I}_{P' \cap Q'}^{Q'} n_{w_G w_Q} = n_{w_G w_Q} \tilde{I}_{P_2}^Q$ where P_2 is a parabolic subgroup corresponding to $w_Q(-\Delta_{P \cap Q})$. Therefore we have

$$R_P \circ I_Q = \tilde{I}_{P_1}^P n_{w_G w_P}^{-1} n_{w_G w_Q} \tilde{R}_{P_2}^Q.$$

Since we have $n_{w_G w_P} n_{w_P w_{P \cap Q}} = n_{w_G w_{P \cap Q}} = n_{w_G w_Q} n_{w_Q w_{P \cap Q}}$, we have $n_{w_G w_P}^{-1} n_{w_G w_Q} = n_{w_P w_{P \cap Q}} n_{w_Q w_{P \cap Q}}^{-1}$. Since $w_P w_{P \cap Q}(\Delta_{P \cap Q}) = \Delta_{P_1}$, we have $\tilde{I}_{P_1}^P n_{w_P w_{P \cap Q}} = \tilde{I}_{P \cap Q}^P$. Similarly we have $n_{w_Q w_{P \cap Q}}^{-1} \tilde{R}_{P_2}^Q = \tilde{R}_{P \cap Q}^Q$. Hence we get Proposition 5.1.

In the rest of this section, we prove Lemma 5.2. Recall a filtration introduced in subsection 3.1. Let $A \subset {}^Q W_0$ be a closed subset and fix a maximal element $w \in A$. Set $A' = A \setminus \{w\}$. Then the quotient

$$\left(\sum_{v \in A} \sigma \otimes T_{n_v}^* \right) / \left(\sum_{v \in A'} \sigma \otimes T_{n_v}^* \right)$$

is isomorphic to σ as vector spaces and the action of $E_{o_-}(\lambda)$ is given by $\tilde{q}(Q, n_w \cdot \lambda) \sigma(E_{o_-, Q}^Q(n_w \cdot \lambda))$ by Lemma 3.5 where $\tilde{q}(Q, \mu) = q(Q', n_{w_G w_Q} \cdot \mu)$. By Remark 3.7, $\tilde{q}(Q, \mu) \neq 1$ if and only if μ is Q -positive.

Lemma 5.3. *Let λ_P^+ be as in Proposition 2.5. Then $n_w \cdot \lambda_P^+$ is Q -positive if and only if $w \in {}^Q W_0 \cap W_{0, P} = {}^{P \cap Q} W_{0, P}$.*

Proof. The element $n_w \cdot \lambda_P^+$ is Q -positive if and only if $\langle \alpha, w \nu(\lambda_P^+) \rangle \leq 0$ for any $\alpha \in \Sigma^+ \setminus \Sigma_Q^+$. Since $\langle \beta, \nu(\lambda_P^+) \rangle \leq 0$ if and only if $\beta \in \Sigma^+ \cup \Sigma_P$, the element $n_w \cdot \lambda_P^+$ is Q -positive if and only if $w^{-1}(\Sigma^+ \setminus \Sigma_Q^+) \subset \Sigma^+ \cup \Sigma_P$. Since $w \in {}^Q W_0$, we have $w^{-1}(\Sigma_Q^+) \subset \Sigma^+$. Hence this is equivalent to $w^{-1}(\Sigma^+) \subset \Sigma^+ \cup \Sigma_P$.

Therefore $w^{-1}(\Sigma^-) \supset \Sigma^- \setminus \Sigma_P^-$ by taking the complement of the both sides. Hence $w(\Sigma^- \setminus \Sigma_P^-) \subset \Sigma^-$. Therefore we have $w \in W_{0,P}$.

For the last part, ${}^QW_0 \cap W_{0,P} \subset {}^{P \cap Q}W_{0,P}$ is obvious. If $w \in W_{0,P}$, then $w^{-1}(\Delta \setminus \Delta_P) \subset \Sigma^+$. Hence if $w \in {}^{P \cap Q}W_{0,P}$, we have $w^{-1}((\Delta \setminus \Delta_P) \cup \Delta_{P \cap Q}) \subset \Sigma^+$. We have $\Delta_Q \subset (\Delta \setminus \Delta_P) \cup \Delta_{P \cap Q}$. Hence $w \in {}^QW_0$. \square

Put $I = \sum_{v \in {}^{P \cap Q}W_{0,P}} \sigma \otimes T_{n_v}^* \subset \tilde{I}_Q(\sigma)$.

Lemma 5.4. $I = \sum_{v \in W_{0,P}} \sigma \otimes T_{n_v}^* = \sum_{v \in W_{0,P}} \sigma \otimes T_{n_v}$.

Proof. For $v \in W_{0,P}$, Take $v_1 \in W_{0,P \cap Q}$, $v_2 \in {}^{P \cap Q}W_{0,P}$ such that $v = v_1 v_2$. Then for any $x \in \sigma$ we have $x \otimes T_{n_v}^* = x T_{n_{v_1}}^{Q*} \otimes T_{n_{v_2}}^*$ since $j_Q^+(T_{n_{v_2}}^{Q*}) = T_{n_{v_2}}^*$ by Corollary 2.7. This gives the first equality. The second equality follows from a usual triangular argument. \square

Lemma 5.5. *The subspace I is stable under the action of $E_{o_-}(w)$ where $w \in W_P(1)$. In particular, I is stable under the action of $j_P^+(\mathcal{H}_P^+)$.*

Proof. We prove $(\sigma \otimes T_{n_v})E_{o_-}(w) \subset I$. Take $a \in W_{0,P}$ and $\lambda \in \Lambda(1)$ such that $w = \lambda n_a$. Then by the Bernstein relations [Vig16, Corollary 5.43], in $\mathcal{H}[q_s^{\pm 1}]$, we have

$$\begin{aligned} T_{n_v} E_{o_-}(w) &\in (C[q_s^{\pm 1}] T_{n_v} E_{o_-}(\lambda) T_{n_a}) \cap \mathcal{H} \\ &\subset \left(\sum_{b \leq v, \mu \in \Lambda(1)} C[q_s^{\pm 1}] E_{o_-}(\mu) T_{n_b} T_{n_a} \right) \cap \mathcal{H}. \end{aligned}$$

Since $v \in W_{0,P}$, $b \leq v$ implies $b \in W_{0,P}$. We have $a \in W_{0,P}$. Hence $T_{n_b} T_{n_a} \in \sum_{c \in W_{0,P}} C[Z_\kappa] T_{n_c}$. Therefore

$$\begin{aligned} T_{n_v} E_{o_-}(w) &\in \left(\sum_{c \in W_{0,P}, \mu \in \Lambda(1)} C[q_s^{\pm 1}] E_{o_-}(\mu) T_{n_c} \right) \cap \mathcal{H} \\ &= \left(\sum_{c \in W_{0,P}, \mu \in \Lambda(1)} C[q_s^{\pm 1}] E_{o_-}(\mu n_c) \right) \cap \mathcal{H} \\ &= \sum_{c \in W_{0,P}, \mu \in \Lambda(1)} C[q_s] E_{o_-}(\mu n_c). \end{aligned}$$

Hence it is sufficient to prove that $\sigma \otimes E_{o_-}(\mu n_c) \subset I$, namely we may assume $v = 1$. Take λ_Q^+ as in Proposition 2.5 such that $\lambda_Q^+ \lambda$ is Q -positive. Then for $x \in \sigma$, we have

$$\begin{aligned} x \otimes E_{o_-}(\lambda n_a) &= x \sigma(E_{o_-,Q}^Q(\lambda_Q^+))^{-1} \otimes E_{o_-}(\lambda_Q^+) E_{o_-}(\lambda n_a) \\ &\in C x \sigma(E_{o_-,Q}^Q(\lambda_Q^+))^{-1} \otimes E_{o_-}(\lambda_Q^+ \lambda n_a). \end{aligned}$$

Here we use the product formula (2.1). Therefore we may assume that λ is Q -positive. Take $a_1 \in W_{0,P \cap Q}$ and $a_2 \in {}^{P \cap Q}W_{0,P}$ such that $a = a_1 a_2$. Then λn_{a_1} is Q -positive. By Lemma 2.20, we have $\ell(\lambda n_a) = \ell(\lambda n_{a_1}) + \ell(n_{a_2})$. Hence by (2.1), we have $E_{o_-}(\lambda n_a) = E_{o_-}(\lambda n_{a_1}) E_{o_-,a_1}(n_{a_2})$. By

Lemma 4.15, we have $E_{o_- \cdot a_1}(n_{a_2}) = E_{o_-}(n_{a_2})$. Hence

$$\begin{aligned} x \otimes E_{o_-}(\lambda n_a) &= x \otimes E_{o_-}(\lambda n_{a_1}) E_{o_-}(n_{a_2}) \\ &= x \sigma(E_{o_-, Q}^Q(\lambda n_{a_1})) \otimes T_{n_{a_2}}. \end{aligned}$$

We get the lemma. \square

Consider the exact sequence

$$0 \rightarrow I \rightarrow I_Q(\sigma) \rightarrow I_Q(\sigma)/I \rightarrow 0$$

as (\mathcal{H}_P^+, j_P^+) -modules. We have the exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}_{(\mathcal{H}_P^+, j_P^+)}(\mathcal{H}_P, I) &\rightarrow \text{Hom}_{(\mathcal{H}_P^+, j_P^+)}(\mathcal{H}_P, I_Q(\sigma)) \\ &\rightarrow \text{Hom}_{(\mathcal{H}_P^+, j_P^+)}(\mathcal{H}_P, I_Q(\sigma)/I). \end{aligned}$$

Lemma 5.6. *Assume that $\bigcap_n p^n \sigma = 0$. Then $\text{Hom}_{(\mathcal{H}_P^+, j_P^+)}(\mathcal{H}_P, I_Q(\sigma)/I) = 0$.*

Proof. Let λ_P^+ as in Proposition 2.5. By Lemma 2.6, $j_P^+(E_{o_-, P}^P(\lambda_P^+)) = E_{o_-}(\lambda_P^+)$. Hence by Proposition 2.5, we have

$$\text{Hom}_{(\mathcal{H}_P^+, j_P^+)}(\mathcal{H}_P, I_Q(\sigma)/I) = \text{Hom}_{C[E_{o_-}(\lambda_P^+)]}(C[E_{o_-}(\lambda_P^+)^{\pm 1}], I_Q(\sigma)/I).$$

Define a $C[E_{o_-}(\lambda_P^+)]$ -module σ_w by

$$\sigma_w(E_{o_-}(\lambda_P^+)) = \tilde{q}(P, n_w \cdot \lambda) e_Q(\sigma)(E_{o_-, Q}^Q(n_w \cdot \lambda_P^+))$$

on the same space as σ . Then $I_Q(\sigma)/I$ has a $C[E_{o_-}(\lambda_P^+)]$ -stable filtration whose subquotient is given by σ_w where $w \in {}^Q W_0 \setminus {}^{P \cap Q} W_{0, P}$. By Lemma 5.3, $\tilde{q}(P, n_w \cdot \lambda) \neq 1$. Hence it is a positive power of p . Therefore the image of $\psi \in \text{Hom}_{C[E_{o_-}(\lambda_P^+)]}(C[E_{o_-}(\lambda_P^+)^{\pm 1}], \sigma_w)$ is contained in $\bigcap_{n \geq 0} p^n \sigma_w$. This is zero by the assumption. \square

Hence to prove Lemma 5.2, it is sufficient to prove

$$(5.1) \quad \text{Hom}_{(\mathcal{H}_P^+, j_P^+)}(\mathcal{H}_P, I) \simeq \text{Hom}_{(\mathcal{H}_{P \cap Q}^{Q+}, j_{P \cap Q}^{Q+})}(\mathcal{H}_{P \cap Q}, \sigma) \otimes_{(\mathcal{H}_{P \cap Q}^{P+}, j_{P \cap Q}^{P+})} \mathcal{H}_P.$$

To construct a homomorphism from the right hand side of (5.1) to the left hand side, it is sufficient to construct an $(\mathcal{H}_{P \cap Q}^{P+}, j_{P \cap Q}^{P+})$ -homomorphism $\text{Hom}_{(\mathcal{H}_{P \cap Q}^{Q+}, j_{P \cap Q}^{Q+})}(\mathcal{H}_{P \cap Q}, \sigma) \rightarrow \text{Hom}_{(\mathcal{H}_P^+, j_P^+)}(\mathcal{H}_P, I)$. Let λ_P^+ as in Proposition 2.5. By Lemma 3.16, we have

$$\text{Hom}_{(\mathcal{H}_P^+, j_P^+)}(\mathcal{H}_P, I)|_{(\mathcal{H}_{P \cap Q}^{P+}, j_{P \cap Q}^{P+})} \simeq \text{Hom}_{(\mathcal{H}_{P \cap Q}^{P+}, j_{P \cap Q}^{P+})}(\mathcal{H}_{P \cap Q}^{P+}, I).$$

(Both sides are isomorphic to $\text{Hom}_{C[E_{o_-}(\lambda_P^+)]}(C[E_{o_-}(\lambda_P^+)^{\pm 1}], I)$.) Now the restriction to from $\mathcal{H}_{P \cap Q}$ to $\mathcal{H}_{P \cap Q}^{P+}$ gives a map $\text{Hom}_{(\mathcal{H}_{P \cap Q}^{Q+}, j_{P \cap Q}^{Q+})}(\mathcal{H}_{P \cap Q}, \sigma) \rightarrow \text{Hom}_{(\mathcal{H}_{P \cap Q}^{P+}, j_{P \cap Q}^{P+})}(\mathcal{H}_{P \cap Q}^{P+}, \sigma)$. Combining the inclusion $\sigma = \sigma \otimes 1 \hookrightarrow I$ which is (\mathcal{H}_Q^+, j_Q^+) -equivariant, we get

$$\begin{aligned} \text{Hom}_{(\mathcal{H}_{P \cap Q}^{Q+}, j_{P \cap Q}^{Q+})}(\mathcal{H}_{P \cap Q}, \sigma) &\rightarrow \text{Hom}_{(\mathcal{H}_{P \cap Q}^{P+}, j_{P \cap Q}^{P+})}(\mathcal{H}_{P \cap Q}^{P+}, I) \\ &\simeq \text{Hom}_{(\mathcal{H}_P^+, j_P^+)}(\mathcal{H}_P, I) \end{aligned}$$

and it gives a homomorphism between (5.1).

We prove that this gives an isomorphism. We have the decomposition

$$I = \bigoplus_{v \in P \cap Q W_{0,P}} \sigma \otimes T_{n_v}.$$

We have $\langle \alpha, \nu(\lambda_P^+) \rangle = 0$ for any $\alpha \in \Delta_P$. Hence the Bernstein relations [Vig16, Corollary 5.43] tells that $T_{n_v} E_{o_-}(\lambda_P^+) = E_{o_-}(n_v \cdot \lambda_P^+) T_{n_v} = E_{o_-}(\lambda_P^+) T_{n_v}$ for any $v \in W_{0,P}$. (For the last part, recall that λ_P^+ is in the center of $W_P(1)$.) Therefore each summand $\sigma \otimes T_{n_v}$ is stable under the action of $E_{o_-}(\lambda_P^+)$ and we have

$$\mathrm{Hom}_{(\mathcal{H}_P^+, j_P^+)}(\mathcal{H}_P, I) \simeq \bigoplus_{v \in P \cap Q W_{0,P}} \mathrm{Hom}_{C[E_{o_-}(\lambda_P^+)]}(C[E_{o_-}(\lambda_P^+)^{\pm 1}], \sigma \otimes T_{n_v}).$$

For $x \in \sigma$, from the above calculation we have $x \otimes T_{n_v} E_{o_-}(\lambda_P^+) = x \otimes E_{o_-}(\lambda_P^+) T_{n_v} = x E_{o_-,Q}^Q(\lambda_P^+) \otimes T_{n_v}$. Hence each summand is isomorphic to $\mathrm{Hom}_{C[E_{o_-,Q}^Q(\lambda_P^+)]}(C[E_{o_-,Q}^Q(\lambda_P^+)^{\pm 1}], \sigma)$. Therefore we have

$$(5.2) \quad \mathrm{Hom}_{(\mathcal{H}_P^+, j_P^+)}(\mathcal{H}_P, I) \simeq \bigoplus_{v \in P \cap Q W_{0,P}} \mathrm{Hom}_{C[E_{o_-,Q}^Q(\lambda_P^+)]}(C[E_{o_-,Q}^Q(\lambda_P^+)^{\pm 1}], \sigma).$$

On the other hand, the right hand side of (5.1) is

$$(5.3) \quad \bigoplus_{v \in P \cap Q W_{0,P \cap Q}} \mathrm{Hom}_{(\mathcal{H}_{P \cap Q}^+, j_{P \cap Q}^+)}(\mathcal{H}_{P \cap Q}, \sigma) \otimes T_{n_v}.$$

Lemma 5.7. *The element λ_P^+ satisfies the condition of $\lambda_{P \cap Q}^{Q+}$ in Proposition 2.5.*

Proof. The element $\lambda_P^+ \in Z(W_P(1))$ satisfies $\langle \alpha, \nu(\lambda_P^+) \rangle < 0$ for any $\alpha \in \Sigma^+ \setminus \Sigma_P^+$. Hence $\langle \alpha, \nu(\lambda_P^+) \rangle < 0$ for any $\alpha \in \Sigma_Q^+ \setminus \Sigma_{P \cap Q}^+$ and $\lambda_P^+ \in Z(W_{P \cap Q}(1))$. \square

Hence we have $\mathcal{H}_{P \cap Q} = \mathcal{H}_{P \cap Q}^{Q+} E_{o_-, P \cap Q}^{P \cap Q}(\lambda_P^+)^{-1}$. Note that, by Proposition 2.6, we have $j_{P \cap Q}^{Q+}(E_{o_-, P \cap Q}^{P \cap Q}(\lambda_P^+)) = E_{o_-, Q}^Q(\lambda_P^+)$. Therefore (5.3) is equal to

$$\bigoplus_{v \in P \cap Q W_{0,P \cap Q}} \mathrm{Hom}_{C[E_{o_-,Q}^Q(\lambda_P^+)]}(C[E_{o_-,Q}^Q(\lambda_P^+)^{\pm 1}], \sigma) \otimes T_{n_v}.$$

This is isomorphic to (5.2). This ends the proof of Lemma 5.2.

Corollary 5.8. *Assume that p is nilpotent in C . Then we have $L_P \circ I_Q \simeq I_P^{P \cap Q} \circ L_{P \cap Q}^P$.*

Proof. If p is nilpotent, then any module satisfies $\bigcap_n p^n \sigma = 0$. Hence we have $R_P \circ I_Q \simeq I_{P \cap Q}^P \circ R_{P \cap Q}^P$. Taking the left adjoint functors of the both sides, we get the corollary. \square

5.3. L_P and Steinberg modules. In this section, we fix a parabolic subgroup $P \subset G$ such that Δ_P and $\Delta \setminus \Delta_P$ are orthogonal to each other. Let σ be an \mathcal{H}_P -module which has the extension $e(\sigma)$ to \mathcal{H} . Then we have an \mathcal{H} -module $\mathrm{St}_Q(\sigma)$ for $Q \supset P$. Let R be another parabolic subgroup.

Lemma 5.9. *We have $L_R(e_G(\sigma)) \simeq e_R(L_{P \cap R}^P(\sigma))$.*

Proof. Let λ_R^- as in Proposition 2.5. Then by Lemma 5.7, this satisfies the condition of $\lambda_{R \cap P}^-$. Hence we have $L_{P \cap R}^P(\sigma) = \sigma E_{o_-, P}^P(\lambda_R^-)^{-1}$. By the definition, λ_R^- is dominant, hence in particular, it is P -negative. Hence $e_G(\sigma)(E_{o_-}(\lambda_R^-)) = e_G(\sigma)(j_P^*(E_{o_-, P}^P(\lambda_R^-))) = \sigma(E_{o_-, P}^P(\lambda_R^-))$ by Lemma 2.6. Therefore, as vector spaces, we have

$$L_R(e_G(\sigma)) = e_G(\sigma)E_{o_-}(\lambda_R^-)^{-1} \simeq \sigma E_{o_-, P}^P(\lambda_R^-)^{-1} = L_{P \cap R}^P(\sigma).$$

Namely, the linear map defined by $e_G(\sigma) \otimes_{(\mathcal{H}_R^-, j_R^{-*})} \mathcal{H}_R \ni x \otimes E_{o_-, R}^R(\lambda_R^-)^{-n} \mapsto x \otimes E_{o_-, P \cap R}^{P \cap R}(\lambda_R^-)^{-n} \in \sigma \otimes_{(\mathcal{H}_{P \cap R}^{P-}, j_{P \cap R}^{P-*})} \mathcal{H}_{P \cap R}$ is an isomorphism.

We prove that this map is $(\mathcal{H}_{P \cap R}^{R-}, j_{P \cap R}^{R-*})$ -equivariant. Let $w \in W_{P \cap R}^R(1)$ and take $k \in \mathbb{Z}_{\geq 0}$ such that $w(\lambda_R^-)^k$ is R -negative. Since the length of λ_R^- as an element of $W_R(1)$ is zero, we have $T_w^{R*} E_{o_-, R}^R(\lambda_R^-)^k = T_w^{R*} T_{(\lambda_R^-)^k}^{R*} = T_{w(\lambda_R^-)^k}^{R*}$. We also have that $E_{o_-, R}^R(\lambda_R^-)$ is in the center of \mathcal{H}_R^- . Replacing R with $P \cap R$, we also have $T_w^{(P \cap R)*} E_{o_-, P \cap R}^{P \cap R}(\lambda_R^-)^k = T_{w(\lambda_R^-)^k}^{(P \cap R)*}$. Hence for $x \in \sigma$, we have

$$\begin{aligned} e_G(\sigma) \otimes_{(\mathcal{H}_R^-, j_R^{-*})} \mathcal{H}_R &\ni x \otimes E_{o_-, R}^R(\lambda_R^-)^{-n} T_w^{R*} \\ &= x \otimes T_w^{R*} E_{o_-, R}^R(\lambda_R^-)^{-n} \\ &= x \otimes T_{w(\lambda_R^-)^k}^{R*} E_{o_-, R}^R(\lambda_R^-)^{-(n+k)} \\ &= x e_G(\sigma)(T_{w(\lambda_R^-)^k}^*) \otimes E_{o_-, R}^R(\lambda_R^-)^{-(n+k)} \\ &= x \sigma(T_{w(\lambda_R^-)^k}^{P*}) \otimes E_{o_-, R}^R(\lambda_R^-)^{-(n+k)} \\ &\mapsto x \sigma(T_{w(\lambda_R^-)^k}^{P*}) \otimes E_{o_-, P \cap R}^{P \cap R}(\lambda_R^-)^{-(n+k)} \\ &= x \otimes T_{w(\lambda_R^-)^k}^{(P \cap R)*} E_{o_-, P \cap R}^{P \cap R}(\lambda_R^-)^{-(n+k)} \\ &= x \otimes E_{o_-, P \cap R}^{P \cap R}(\lambda_R^-)^{-n} T_w^{(P \cap R)*} \\ &\in \sigma \otimes_{(\mathcal{H}_{P \cap R}^{P-}, j_{P \cap R}^{P-*})} \mathcal{H}_{P \cap R}. \end{aligned}$$

Therefore the above homomorphism is $(\mathcal{H}_{P \cap R}^{R-}, j_{P \cap R}^{R-*})$ -equivariant.

Let P_2 be a parabolic subgroup corresponding to $\Delta \setminus \Delta_P$. Fix $w \in W_{P_2 \cap R, \text{aff}}(1)$ and take $v \in W_0$ and $\lambda \in \Lambda(1)$ such that $w = n_v \lambda$. Then we have $\nu(\lambda) \in \mathbb{R}(\Delta_{P_2}^\vee \cap \Delta_R^\vee)$. In particular, $\langle \alpha, \nu(\lambda) \rangle = 0$ for any $\alpha \in \Delta_P$. Take a dominant $\mu \in Z(W_R(1)) \cap W_{P_2, \text{aff}}(1)$ such that $\langle \alpha, \nu(\mu) \rangle = 0$ for any $\alpha \in \Delta_R \cup \Delta_P$ and $\langle \alpha, \nu(\mu) \rangle$ is sufficiently large for $\alpha \in \Delta \setminus (\Delta_R \cup \Delta_P) = \Delta_{P_2} \setminus \Delta_R$. Then $\langle \alpha, \nu(\lambda\mu) \rangle = 0$ for any $\alpha \in \Delta_P$. For $\alpha \in \Sigma_{P_2}^+ \setminus \Sigma_R^+$, we can take μ such that $\langle \alpha, \nu(\lambda\mu) \rangle \geq 0$. For such μ , we have $\langle \alpha, \nu(\lambda\mu) \rangle \geq 0$ for any $\alpha \in \Sigma_P^+ \cup (\Sigma_{P_2}^+ \setminus \Sigma_R^+) = \Sigma^+ \setminus \Sigma_R^+$. Namely we can take μ such that $\lambda\mu$ is R -negative. Since $w\mu$ is R -negative, for $x \in \sigma$, we have

$$x \otimes E_{o_-, R}^R(\lambda_R^-)^{-n} T_{w\mu}^{R*} = x e_G(\sigma)(T_{w\mu}^*) \otimes E_{o_-, R}^R(\lambda_R^-)^{-n} \in e_G(\sigma) \otimes_{(\mathcal{H}_R^-, j_R^{-*})} \mathcal{H}_R.$$

Since $w\mu \in W_{P_2, \text{aff}}(1)$, $e_G(\sigma)(T_{w\mu}^*) = 1$. Hence $L_R(e_G(\sigma))(T_{w\mu}^{R*}) = 1$. In particular, by taking $w = 1$, we have $L_R(e_G(\sigma))(T_\mu^{R*}) = 1$. Since $\mu \in$

$Z(W_R(1))$, we have $T_{w\mu}^{R*} = T_w^{R*} T_\mu^{R*}$. Therefore we get $L_R(e_G(\sigma))(T_w^{R*}) = 1$. By the characterization of $e_R(L_{P \cap R}^P(\sigma))$, we get the lemma. \square

Proposition 5.10. *Assume that p is nilpotent in C . We have*

$$L_R(\text{St}_Q(\sigma)) = \begin{cases} \text{St}_{Q \cap R}^R(L_{P \cap R}^P(\sigma)) & (\Delta_Q \cup \Delta_R = \Delta), \\ 0 & (\text{otherwise}). \end{cases}$$

Proof. We have an exact sequence

$$\bigoplus_{Q' \supseteq Q} I_{Q'}(e_{Q'}(\sigma)) \rightarrow I_Q(e_Q(\sigma)) \rightarrow \text{St}_Q(\sigma) \rightarrow 0.$$

Since L_R is exact, we have

$$\bigoplus_{Q' \supseteq Q} L_R(I_{Q'}(e_{Q'}(\sigma))) \rightarrow L_R(I_Q(e_Q(\sigma))) \rightarrow L_R(\text{St}_Q(\sigma)) \rightarrow 0.$$

By Corollary 5.8, we have

$$\bigoplus_{Q' \supseteq Q} I_{Q' \cap R}^R(L_{Q' \cap R}^{Q'}(e_{Q'}(\sigma))) \rightarrow I_{Q \cap R}^R(L_{Q \cap R}^Q(e_Q(\sigma))) \rightarrow L_R(\text{St}_Q(\sigma)) \rightarrow 0.$$

By Lemma 5.9, we have

$$\bigoplus_{Q' \supseteq Q} I_{Q' \cap R}^R(e_{Q' \cap R}(L_{P \cap R}^P(\sigma))) \rightarrow I_{Q \cap R}^R(e_{Q \cap R}(L_{P \cap R}^P(\sigma))) \rightarrow L_R(\text{St}_Q(\sigma)) \rightarrow 0.$$

Hence if there exists $Q' \supseteq Q$ such that $Q' \cap R = Q \cap R$, then $L_R(\text{St}_Q(\sigma)) = 0$. Such Q' exists if and only if $\Delta_Q \cup \Delta_R \neq \Delta$. Therefore, if $\Delta_Q \cup \Delta_R \neq \Delta$, then $L_R(\text{St}_Q(\sigma)) = 0$. If $\Delta_Q \cup \Delta_R = \Delta$ then $\{Q' \cap R \mid Q' \supseteq Q\} = \{Q'' \mid R \supset Q'' \supseteq Q \cap R\}$. Therefore we get $L_R(\text{St}_Q(\sigma)) \simeq \text{St}_{Q \cap R}^R(L_{P \cap R}^P(\sigma))$. \square

5.4. R_P and an exactness. In the next subsection, we will prove the following proposition.

Proposition 5.11. *Let P be a parabolic subgroup, σ an \mathcal{H}_P -module which has the extension to \mathcal{H} and Q a parabolic subgroup containing P . Assume that $\bigcap_{n \in \mathbb{Z}_{\geq 0}} p^n \sigma = 0$. Then for any parabolic subgroup R , we have*

$$R_R(\text{St}_Q(\sigma)) = \begin{cases} \text{St}_{Q \cap R}^R(R_{P \cap R}^P(\sigma)) & (\Delta_Q = \Delta_{Q \cap R} \cup \Delta_P), \\ 0 & (\text{otherwise}). \end{cases}$$

The proof of this proposition is similar to that of Proposition 5.10. However, to use that argument, we need the following lemma. Note that this is not obvious since R_R is not right exact.

Lemma 5.12. *Let P be a parabolic subgroup, σ an \mathcal{H}_P -module which has the extension to \mathcal{H} and Q a parabolic subgroup containing P . Then for any parabolic subgroup R , the sequence*

$$\bigoplus_{Q_1 \supseteq Q} R_R(I_{Q_1}(e_{Q_1}(\sigma))) \rightarrow R_R(I_Q(e_Q(\sigma))) \rightarrow R_R(\text{St}_Q(\sigma)) \rightarrow 0$$

is exact.

We remark that we do not assume that $\bigcap_n p^n \sigma = 0$. The aim of this subsection is to prove this lemma.

Lemma 5.13. *To prove Lemma 5.12, we may assume $\sigma = \mathbf{1}$ and R contains P .*

Proof. Put $R' = n_{w_G w_R} R^{\text{op}} n_{w_G w_R}^{-1}$. Let R'_1 (resp. R'_2) be a parabolic subgroup corresponding to $\Delta_{R'} \cup \Delta_P$ (resp. $\Delta_{R'} \cup \Delta_{P_2}$). Let $\lambda_1 = \lambda_{R'_1}^+$ and $\lambda_2 = \lambda_{R'_2}^+$ be as in Proposition 2.5. Moreover we take λ_1 from $W_{P_2, \text{aff}}(1)$ and λ_2 from $W_{P, \text{aff}}(1)$. Then $\lambda_1 \lambda_2$ satisfies the condition of $\lambda_{R'}^+$. We have

$$R_R(\pi) = n_{w_G w_R}^{-1} \text{Hom}_{(\mathcal{H}_{R'}^+, j_{R'}^+)}(\mathcal{H}_{R'}, \pi).$$

Since $\mathcal{H}_{R'} = \mathcal{H}_{R'}(T_{\lambda_{R'}^+}^{R'})^{-1}$, we have

$$R_R(\pi) \simeq \text{Hom}_{C[T_{\lambda_{R'}^+}]}(C[(T_{\lambda_{R'}^+})^{\pm 1}], \pi)$$

as vector spaces. Therefore we have

$$R_R(\pi) \simeq \text{Hom}_{C[T_{\lambda_1}]}(C[(T_{\lambda_1})^{\pm 1}], \text{Hom}_{C[T_{\lambda_2}]}(C[(T_{\lambda_2})^{\pm 1}], \pi)).$$

Take $\pi = I_Q(e_Q(\sigma))$. Then $I_Q(e_Q(\sigma)) \simeq I_Q(\mathbf{1}) \otimes e_G(\sigma)$ by Lemma 3.17 and $(x_1 \otimes x_2)T_{\lambda_1} = x_1 T_{\lambda_1} \otimes x_2$ and $(x_1 \otimes x_2)T_{\lambda_2} = x_1 \otimes x_2 T_{\lambda_2}$ for $x_1 \in I_Q(\mathbf{1})$ and $x_2 \in e_G(\sigma)$ by Remark 3.14. Hence we get

$$\begin{aligned} & R_R(I_Q(e_Q(\sigma))) \\ & \simeq \text{Hom}_{C[T_{\lambda_1}]}(C[(T_{\lambda_1})^{\pm 1}], I_Q(\mathbf{1})) \otimes \text{Hom}_{C[T_{\lambda_2}]}(C[(T_{\lambda_2})^{\pm 1}], e_G(\sigma)). \end{aligned}$$

Since we also have the same formula for $I_{Q_1}(e_{Q_1}(\sigma))$ where $Q_1 \supsetneq Q$ and $\text{St}_Q \sigma$, the sequence in Lemma 5.12 is equal to the sequence

$$\begin{array}{c} \bigoplus_{Q_1 \supsetneq Q} \text{Hom}_{C[T_{\lambda_1}]}(C[(T_{\lambda_1})^{\pm 1}], I_{Q_1}(\mathbf{1})) \otimes \text{Hom}_{C[T_{\lambda_2}]}(C[(T_{\lambda_2})^{\pm 1}], e_G(\sigma)) \\ \downarrow \\ \text{Hom}_{C[T_{\lambda_1}]}(C[(T_{\lambda_1})^{\pm 1}], I_Q(\mathbf{1})) \otimes \text{Hom}_{C[T_{\lambda_2}]}(C[(T_{\lambda_2})^{\pm 1}], e_G(\sigma)) \\ \downarrow \\ \text{Hom}_{C[T_{\lambda_1}]}(C[(T_{\lambda_1})^{\pm 1}], \text{St}_Q(\mathbf{1})) \otimes \text{Hom}_{C[T_{\lambda_2}]}(C[(T_{\lambda_2})^{\pm 1}], e_G(\sigma)) \\ \downarrow \\ 0 \end{array}$$

Hence it is sufficient to prove that the exact sequence $\bigoplus_{Q_1 \supsetneq Q} I_{Q_1}(\mathbf{1}) \rightarrow I_Q(\mathbf{1}) \rightarrow \text{St}_Q(\mathbf{1}) \rightarrow 0$ is still exact after applying $\text{Hom}_{C[T_{\lambda_1}]}(C[(T_{\lambda_1})^{\pm 1}], \cdot)$. Let R_1 be a parabolic subgroup corresponding to $\Delta_R \cup \Delta_P$. Then we have $R'_1 = n_{w_G w_{R_1}} R_1^{\text{op}} n_{w_G w_{R_1}}^{-1}$ and $\text{Hom}_{C[T_{\lambda_1}]}(C[(T_{\lambda_1})^{\pm 1}], \cdot) \simeq R_{R_1}$ as vector spaces. Hence we may assume $\sigma = \mathbf{1}$ and $R = R_1$, namely R contains P . \square

We prove Lemma 5.12 assuming $\sigma = \mathbf{1}$ and R contains P . Let $A \subset W_0^Q$ be an open subset and $w \in A$ a minimal element and $I_Q(\mathbf{1})_A$ a filtration defined in subsection 3.1. Set $A' = A \setminus \{w\}$. As in Remark 3.9, put $I_{Q_1, A} =$

$I_{Q_1}(\mathbf{1}) \cap I_Q(\mathbf{1})_A$ and let $\text{St}_{Q,A}$ be the image of $I_Q(\mathbf{1})_A$ in $\text{St}_Q(\mathbf{1})$. Then we have a commutative diagram with exact rows and columns (Remark 3.9):

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 \oplus_{Q_1 \supsetneq Q} I_{Q_1,A'} & \longrightarrow & I_{Q,A'} & \longrightarrow & \text{St}_{Q,A'} & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \oplus_{Q_1 \supsetneq Q} I_{Q_1,A} & \longrightarrow & I_{Q,A} & \longrightarrow & \text{St}_{Q,A} & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \oplus_{Q_1 \supsetneq Q} I_{Q_1,A}/I_{Q_1,A'} & \longrightarrow & I_{Q,A}/I_{Q,A'} & \longrightarrow & \text{St}_{Q,A}/\text{St}_{Q,A'} & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & & 0 & & 0 & &
 \end{array}$$

It is sufficient to prove that this diagram remains exact after applying R_R . By induction on $\#A$, it is sufficient to prove that R_R preserves the exactness of the following three sequences.

$$(5.4) \quad \bigoplus_{Q_1 \supsetneq Q} I_{Q_1,A}/I_{Q_1,A'} \rightarrow I_{Q,A}/I_{Q,A'} \rightarrow \text{St}_{Q,A}/\text{St}_{Q,A'} \rightarrow 0,$$

$$(5.5) \quad 0 \rightarrow I_{Q_1,A'} \rightarrow I_{Q_1,A} \rightarrow I_{Q_1,A}/I_{Q_1,A'} \rightarrow 0 \quad (Q_1 \supset Q),$$

$$(5.6) \quad 0 \rightarrow \text{St}_{Q,A'} \rightarrow \text{St}_{Q,A} \rightarrow \text{St}_{Q,A}/\text{St}_{Q,A'} \rightarrow 0.$$

First we prove that R_R preserves the exactness of (5.4). Assume that $w \notin W_0^{Q_1}$ for any $Q_1 \supsetneq Q$. Then $\bigoplus_{Q_1 \supsetneq Q} I_{Q_1,A}/I_{Q_1,A'} = 0$ by Lemma 3.10. Hence $I_{Q,A}/I_{Q,A'} \xrightarrow{\sim} \text{St}_{Q,A}/\text{St}_{Q,A'}$. Therefore R_R preserves the exactness of (5.4). If $w \in W_0^{Q_1}$ for some $Q_1 \supsetneq Q$, then $I_{Q_1,A}/I_{Q_1,A'} \simeq I_{Q,A}/I_{Q,A'}$ for such Q_1 by Lemma 3.10. Hence $\text{St}_{Q,A}/\text{St}_{Q,A'} = 0$. Moreover, fix $Q_0 \supsetneq Q$ such that $w \in W_0^{Q_0}$. Then $I_{Q,A}/I_{Q,A'} \simeq I_{Q_0,A}/I_{Q_0,A'} \hookrightarrow \bigoplus_{Q_1 \supsetneq Q} I_{Q_1,A}/I_{Q_1,A'}$ is a splitting of $\bigoplus_{Q_1 \supsetneq Q} I_{Q_1,A}/I_{Q_1,A'} \rightarrow I_{Q,A}/I_{Q,A'}$. Hence (5.4) is exact after applying R_R .

Set $R' = n_{w_G w_R} R^{\text{op}} n_{w_G w_R}^{-1}$ and let $\lambda_{R'}^+$ be as in Proposition 2.5. To prove that R_R preserves the exactness of (5.5) and (5.6), we analyze the action of $X = T_{\lambda_{R'}^+}$ on $I_{Q_1,A}/I_{Q_1,A'}$ for $Q_1 \supset Q$.

Lemma 5.14. *The action of $X = T_{\lambda_{R'}^+}$ on $I_{Q_1,A}/I_{Q_1,A'}$ is a power of p and it is 1 if and only if $w_G(\Sigma_Q) \subset \Sigma_{R'}$ and $w \in W_{0,R'} w_G w_Q$.*

Proof. We may assume $Q_1 = Q$ by Lemma 3.10. Note that since $\lambda_{R'}^+$ is anti-dominant, we have $E_{o_-}(\lambda_{R'}^+) = T_{\lambda_{R'}^+}$. Hence by Proposition 3.2, the action of $T_{\lambda_{R'}^+} = E_{o_-}(\lambda_{R'}^+)$ is given by $q(Q, n_w^{-1} \cdot \lambda_{R'}^+) \mathbf{1}(E_{o_-}^Q(n_w^{-1} \cdot \lambda_{R'}^+))$. Since $w \in W_0^Q$, $w(\Sigma_Q^+) \subset \Sigma^+$. Therefore $n_w^{-1} \cdot \lambda_{R'}^+$ is anti-dominant with respect to Σ_Q^+ since $\lambda_{R'}^+$ is anti-dominant with respect to Σ^+ . Hence $\mathbf{1}(E_{o_-}^Q(n_w^{-1} \cdot \lambda_{R'}^+)) = \mathbf{1}(T_{n_w^{-1} \cdot \lambda_{R'}^+}^Q) = q_{n_w^{-1} \cdot \lambda_{R'}^+, Q}$. Therefore the action of $T_{\lambda_{R'}^+}$ on this subquotient

is given by a power of p and it is 1 if and only if $q(Q, n_w^{-1} \cdot \lambda_{R'}^+) = 1$ and $\ell_Q(n_w^{-1} \cdot \lambda_{R'}^+) = 0$.

Put $Q' = n_{w_G w_Q} Q^{\text{op}} n_{w_G w_Q}^{-1}$. Note that $n_w^{-1} \cdot \lambda_{R'}^+$ is Q -negative if and only if $(n_{w_G w_Q} n_w^{-1}) \cdot \lambda_{R'}^+$ is Q' -positive since $(w_G w_Q)^{-1}(\Sigma^+ \setminus \Sigma_{Q'}^+) = \Sigma^- \setminus \Sigma_Q^-$. Therefore $n_w^{-1} \cdot \lambda_{R'}^+$ is Q -negative if and only if $w_G w_Q w^{-1} \in W_{0, R'}$ by Lemma 5.3, namely $w \in W_{0, R'} w_G w_Q$.

The length of $n_w^{-1} \cdot \lambda_{R'}^+ \in W_Q(1)$ is 0 if and only if $\langle \alpha, \nu(n_w^{-1} \cdot \lambda_{R'}^+) \rangle = 0$ for any $\alpha \in \Sigma_Q$. Since $\langle \beta, \nu(\lambda_{R'}^+) \rangle = 0$ if and only if $\beta \in \Sigma_{R'}$, the length of $\lambda_{R'}^+ \in W_Q(1)$ is 0 if and only if $w(\Sigma_Q) \subset \Sigma_{R'}$. Since we have $w \in W_{0, R'} w_G w_Q$, $w(\Sigma_Q) \subset \Sigma_{R'}$ if and only if $w_G(\Sigma_Q) \subset \Sigma_{R'}$. \square

The subset $W_{0, R'} \cap {}^{Q'}W_0$ is closed in ${}^{Q'}W_0$. Hence by Proposition 2.21, $W_{0, R'} w_G w_Q \cap W_0^Q$ is open in W_0^Q . Therefore the exactness of (5.5) follows from the following general lemma. In the following lemma, we call a subset A of a partially ordered set open if $a \in A$ and $b \geq a$ implies $b \in A$.

Lemma 5.15. *Let Γ be a partially ordered set and M a $C[X]$ -module with the decomposition into C -submodules $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$. Assume the following.*

- *For each open subset $\Delta \subset \Gamma$, $M_\Delta = \bigoplus_{\gamma \in \Delta} M_\gamma$ is $C[X]$ -stable.*
- *For each open subset $\Delta \subset \Gamma$ and a minimal element $\gamma \in \Delta$, the action of X on $M_\Delta / M_{\Delta \setminus \{\gamma\}}$ is given by p^{n_γ} for some $n_\gamma \in \mathbb{Z}_{\geq 0}$.*
- *The subset $\Gamma_0 = \{\gamma \in \Gamma \mid n_\gamma = 0\}$ is open.*

Then for each open subset $\Delta \subset \Gamma$ and a minimal element $\gamma \in \Delta$, the homomorphism

$$\text{Hom}_{C[X]}(C[X^{\pm 1}], M_\Delta) \rightarrow \text{Hom}_{C[X]}(C[X^{\pm 1}], M_\Delta / M_{\Delta \setminus \{\gamma\}})$$

is surjective.

Proof. Put $\Delta_0 = \Delta \cap \Gamma_0$. We divide the map into two maps:

$$(5.7) \quad \text{Hom}_{C[X]}(C[X^{\pm 1}], M_\Delta) \rightarrow \text{Hom}_{C[X]}(C[X^{\pm 1}], M_\Delta / M_{\Delta_0 \setminus \{\gamma\}}),$$

$$(5.8) \quad \text{Hom}_{C[X]}(C[X^{\pm 1}], M_\Delta / M_{\Delta_0 \setminus \{\gamma\}}) \rightarrow \text{Hom}_{C[X]}(C[X^{\pm 1}], M_\Delta / M_{\Delta \setminus \{\gamma\}}).$$

We prove that both maps are surjective.

(1) We prove the surjectivity of (5.7). Since $M_{\Delta_0 \setminus \{\gamma\}}$ have a filtration such that X is invertible on successive quotients, X is also invertible on $M_{\Delta_0 \setminus \{\gamma\}}$. Hence the claim follows from the following claim.

Claim. Let N be a $C[X]$ -module and assume that X is invertible on N . Then $\text{Ext}_{C[X]}^1(C[X^{\pm 1}], N) = 0$.

Proof of Claim. Let $0 \rightarrow N \rightarrow L \rightarrow C[X^{\pm 1}] \rightarrow 0$ be an exact sequence of $C[X]$ -modules. Since X is invertible on N and $C[X^{\pm 1}]$, X is also invertible on L . Namely $0 \rightarrow N \rightarrow L \rightarrow C[X^{\pm 1}] \rightarrow 0$ is also an exact sequence of $C[X^{\pm 1}]$ -modules. Hence $\text{Ext}_{C[X]}^1(C[X^{\pm 1}], N) = \text{Ext}_{C[X^{\pm 1}]}^1(C[X^{\pm 1}], N)$. This is obviously zero. \square

(2) Next we prove the surjectivity of (5.8) assuming $\gamma \in \Gamma_0$. We have $M_\Delta/M_{\Delta_0 \setminus \{\gamma\}} \supset M_{\Delta_0}/M_{\Delta_0 \setminus \{\gamma\}} \simeq M_\gamma \simeq M_\Delta/M_{\Delta \setminus \{\gamma\}}$. Hence from the following diagram

$$\begin{array}{ccc} \mathrm{Hom}_{C[X]}(C[X^{\pm 1}], M_{\Delta_0}/M_{\Delta_0 \setminus \{\gamma\}}) & & \\ \downarrow & \searrow \sim & \\ \mathrm{Hom}_{C[X]}(C[X^{\pm 1}], M_\Delta/M_{\Delta_0 \setminus \{\gamma\}}) & \longrightarrow & \mathrm{Hom}_{C[X]}(C[X^{\pm 1}], M_\Delta/M_{\Delta \setminus \{\gamma\}}), \end{array}$$

(5.8) is surjective in this case.

(3) Finally we prove the surjectivity of (5.8) assuming $\gamma \notin \Gamma_0$. Note that in general we have $\mathrm{Hom}_{C[X]}(C[X^{\pm 1}], N) = \{(m_n)_{n \in \mathbb{Z}_{\geq 0}} \mid m_n \in N, X m_n = m_{n-1}\}$ by $\varphi \mapsto (\varphi(X^{-n}))$. Recall that the action of X on $M_\Delta/M_{\Delta \setminus \{\gamma\}}$ is given by p^{n_γ} with $n_\gamma > 0$. (We have assumed that $\gamma \notin \Gamma_0$.) Hence to give an element in $\mathrm{Hom}_{C[X]}(C[X^{\pm 1}], M_\Delta/M_{\Delta \setminus \{\gamma\}})$ is equivalent to give a sequence of elements $(m_n^{(\gamma)})$ in M_γ such that $p^{n_\gamma} m_n^{(\gamma)} = m_{n-1}^{(\gamma)}$. We prove that we can extend this element to $m_n = (m_n^{(\delta)})_{\delta \in \Delta, n \in \mathbb{Z}_{\geq 0}}$ in $M_\Delta = \bigoplus_{\delta \in \Delta} M_\delta$ such that $X m_n = m_{n-1}$. Since $p^{n_\gamma} m_n^{(\gamma)} = m_{n-1}^{(\gamma)}$ with $n_\gamma > 0$, we have $m_n^{(\gamma)} \in \bigcap_k p^k M_\gamma$. We prove that we can take an extension $m_n = (m_n^{(\delta)})$ from $\bigcap_k p^k M_\Delta$.

According to the decomposition $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$, we have a linear map $X_{\gamma_1, \gamma_2} \in \mathrm{Hom}_C(M_{\gamma_2}, M_{\gamma_1})$ such that $X m_\gamma = \sum_{\gamma' \in \Gamma} X_{\gamma', \gamma} m_\gamma$ for $m_\gamma \in M_\gamma \subset M$. By the assumption $X_{\gamma_1, \gamma_2} = 0$ if $\gamma_1 \not\geq \gamma_2$. We also have $X_{\gamma', \gamma'} = p^{n_{\gamma'}}$ with $n_{\gamma'} \geq 0$. The condition $X m_n = m_{n-1}$ is equivalent to

$$(5.9) \quad m_{n-1}^{(\delta_1)} = \sum_{\delta_2 \leq \delta_1} X_{\delta_1, \delta_2} m_n^{(\delta_2)}.$$

We prove the existence of $m_n = (m_n^{(\delta)})$ using a triangular argument, namely we take $m_n^{(\delta)}$ which satisfies (5.9) inductively on δ .

Let $\delta \in \Delta \setminus \{\gamma\}$ and assume that we have taken $m_n^{(\delta')}$ for $\delta' \in \Delta$ such that $\delta' < \delta$. Since $m_n^{(\delta')} \in \bigcap_k p^k M_{\delta'}$ for any $\delta' < \delta$, we can take $x_n^{(\delta')} \in \bigcap_k p^k M_{\delta'}$ such that $p^{n_\delta} x_n^{(\delta')} = m_n^{(\delta')}$. We also have $x_{n-1}^{(\delta)} \in \bigcap_k p^k M_\delta$ such that $p^{n_\delta} x_{n-1}^{(\delta)} = m_{n-1}^{(\delta)}$. Then define $m_n^{(\delta)} = x_{n-1}^{(\delta)} - \sum_{\delta' < \delta} X_{\delta, \delta'} x_n^{(\delta')}$ and it satisfies $m_{n-1}^{(\delta)} = p^{n_\delta} m_n^{(\delta)} + \sum_{\delta' < \delta} X_{\delta, \delta'} m_n^{(\delta')}$. Since $X_{\delta, \delta} = p^{n_\delta}$, it means that (5.9) holds for $\delta_1 = \delta$. \square

For (5.6), we use the following lemma with Lemma 5.15.

Lemma 5.16. *Set $B = W_0^Q \setminus \bigcup_{Q_1 \supsetneq Q} W_0^{Q_1}$. For $w \in B$, we define $\mathrm{St}_{Q,w} \subset \mathrm{St}_Q(\mathbf{1})$ by the image of*

$$\{\varphi \in I_Q(\mathbf{1}) \mid \varphi(T_{n_v}) = 0 \ (v \in W_0^Q \setminus \{w\})\} \hookrightarrow I_Q(\mathbf{1}) \rightarrow \mathrm{St}_Q(\mathbf{1}).$$

Then we have $\mathrm{St}_Q(\mathbf{1}) = \bigoplus_{w \in B} \mathrm{St}_{Q,w}$ and for any open $A \subset W_0^Q$, we have $\bigoplus_{w \in A \cap B} \mathrm{St}_{Q,w} = \mathrm{St}_{Q,A}$.

Proof. We prove $\sum_{v \in A \cap B} \mathrm{St}_{Q,v} = \mathrm{St}_{Q,A}$ by induction on $\#A$. Let $w \in A$ be a minimal element and set $A' = A \setminus \{w\}$. If $w \notin B$ then $A \cap B = A' \cap B$ and

$\text{St}_{Q,A} = \text{St}_{Q,A'}$ by Lemma 3.11. If $w \in B$, then $\text{St}_{Q,A}/\text{St}_{Q,A'} \simeq I_{Q,A}/I_{Q,A'} \simeq \{\varphi \in I_Q(\mathbf{1}) \mid \varphi(T_{n_v}) = 0 \ (v \in W_0^Q \setminus \{w\})\}$. Therefore $\text{St}_{Q,A'} + \text{St}_{Q,w} = \text{St}_{Q,A}$. By inductive hypothesis, $\text{St}_{Q,A'} = \sum_{v \in A' \cap B} \text{St}_{Q,v}$. We get $\sum_{v \in A \cap B} \text{St}_{Q,v} = \text{St}_{Q,A}$.

Let $x_w \in \text{St}_{Q,w}$ ($w \in B$) such that $\sum_{w \in B} x_w = 0$. Assume that there exists $w \in B$ such that $x_w \neq 0$ and assume that w is minimal subject to $x_w \neq 0$. Put $A' = \{v \in W_0^Q \mid v \geq v_1 \text{ for some } v_1 \in W_0^Q \setminus \{w\} \text{ such that } x_{v_1} \neq 0\}$. Then A' is open and $w \notin A'$. Hence $\sum_{v \neq w} x_v \in \text{St}_{Q,A'}$. Set $A = A' \cup \{w\}$. Let $y_w \in I_Q(e_Q(\sigma))$ such that $y_w(T_{n_v}) = 0$ for $v \in W_0^Q \setminus \{w\}$ and the image of y_w in $\text{St}_{Q,w}$ is x_w . Then the image of y_w under

$$\{\varphi \in I_Q(\mathbf{1}) \mid \varphi(T_{n_v}) = 0 \ (v \in W_0^Q \setminus \{w\})\} \simeq I_{Q,A}/I_{Q,A'} \simeq \text{St}_{Q,A}/\text{St}_{Q,A'}$$

is equal to the image of x_w . Here the last isomorphism is by Lemma 3.11. Since $\sum_{v \neq w} x_v \in \text{St}_{Q,A'}$, it is equal to the image of $\sum_{v \in B} x_v$ by $\text{St}_{Q,A} \rightarrow \text{St}_{Q,A}/\text{St}_{Q,A'}$. Since $\sum_{v \in B} x_v = 0$, we have $y_w = 0$. This contradicts to $x_w \neq 0$. \square

5.5. R_P and Steinberg modules. As in the previous subsection, let P be a parabolic subgroup and σ an \mathcal{H}_P -module which has the extension to \mathcal{H} . We prove Proposition 5.11 in this subsection. As in the case of L_R , we start with the following lemma.

Lemma 5.17. *Assume that $\bigcap_{n \in \mathbb{Z}_{\geq 0}} p^n \sigma = 0$. We have*

$$R_R(e_G(\sigma)) = \begin{cases} e_R(R_{R \cap P}^P(\sigma)) & (\Delta = \Delta_R \cup \Delta_P), \\ 0 & (\text{otherwise}). \end{cases}$$

Proof. First assume that $\Delta \neq \Delta_R \cup \Delta_P$ and we prove $R_R(e_G(\sigma)) = 0$. Let R_1 be a parabolic subgroup corresponding to $\Delta_R \cup \Delta_P$. It is sufficient to prove that $R_{R_1}(e_G(\sigma)) = 0$. Take $\lambda_{R_1}^+$ as in Proposition 2.5 for R_1' where $R_1' = n_{w_G w_{R_1}} R_1^{\text{op}} n_{w_G w_{R_1}}^{-1}$. Let P_2 be a parabolic subgroup corresponding to $\Delta \setminus \Delta_P$. Since $R_1' \neq G$ and $P \subset R_1'$, there exists $\alpha \in \Delta \setminus \Delta_{R_1'} = \Delta_{P_2} \setminus \Delta_{R_1'}$ and for such α , we have $\langle \alpha, \nu(\lambda_{R_1'}^+) \rangle < 0$. Hence the length of $\lambda_{R_1'}^+$ as an element of $W_{P_2}(1)$ is positive. Therefore $e_G(\sigma)(T_{\lambda_{R_1'}^+})$ is divided by p . Hence, the image of any $\varphi \in \text{Hom}_{(\mathcal{H}_{R_1'}^+, j_{R_1'}^+)}(\mathcal{H}_{R_1'}, e_G(\sigma)) = \text{Hom}_{C[T_{\lambda_{R_1'}^+}]}(C[(T_{\lambda_{R_1'}^+})^{\pm 1}], e_G(\sigma))$ is in $\bigcap_{n \in \mathbb{Z}_{\geq 0}} p^n e_G(\sigma) = 0$. (As C -modules, we have $e_G(\sigma) = \sigma$.) We get $R_{R_1}(e_G(\sigma)) = 0$.

Now assume that $\Delta_R \cup \Delta_P = \Delta$, or in other words, $\Delta_{P_2} \subset \Delta_R$. Since $\Delta = \Delta_P \cup \Delta_{P_2}$ and $\Delta_R = \Delta_{R \cap P} \cup \Delta_{P_2}$ are orthogonal decompositions, we have decompositions $W_0 = W_{0,P} \times W_{0,P_2}$ and $W_{0,R} = W_{0,P \cap R} \times W_{0,P_2}$ as Coxeter groups. Hence we have $w_G = w_P w_{P_2}$ and $w_R = w_{P \cap R} w_{P_2}$. We have $w_G w_R = w_P w_{P \cap R}$. Set $R' = n_{w_G w_R} R^{\text{op}} n_{w_G w_R}^{-1}$. We also have $P \cap R' = n_{w_P w_{P \cap R}} (P \cap R)^{\text{op}} n_{w_P w_{P \cap R}}^{-1}$. By the definition of the right adjoint functors, we have

$$\begin{aligned} R_R(e_G(\sigma)) &= n_{w_G w_R}^{-1} \text{Hom}_{(\mathcal{H}_{R'}^+, j_{R'}^+)}(\mathcal{H}_{R'}, e_G(\sigma)), \\ R_{P \cap R}^P(\sigma) &= n_{w_P w_{P \cap R}}^{-1} \text{Hom}_{(\mathcal{H}_{P \cap R'}^{P+}, j_{P \cap R'}^{P+})}(\mathcal{H}_{P \cap R'}, \sigma) \end{aligned}$$

Since $w_G w_R = w_P w_{P \cap R}$ from the assumption, replacing R' with R , with Lemma 2.27, it is sufficient to prove that $A = \text{Hom}_{(\mathcal{H}_R^+, j_R^+)}(\mathcal{H}_R, e_G(\sigma))$ is isomorphic to $e_R(B)$ where $B = \text{Hom}_{(\mathcal{H}_{P \cap R}^{P+}, j_{P \cap R}^{P+})}(\mathcal{H}_{P \cap R}, \sigma)$.

The map $\varphi \mapsto (\varphi((T_{\lambda_R^+}^R)^{-n}))$ gives an isomorphism

$$A \simeq \{(x_n)_{n \in \mathbb{Z}_{\geq 0}} \mid x_{n+1} e_G(\sigma)(T_{\lambda_R^+}) = x_n, x_n \in e_G(\sigma)\}$$

Since λ_R^+ is anti-dominant, $T_{\lambda_R^+} = E_{o_-}(\lambda_R^+)$ and $T_{\lambda_R^+}^P = E_{o_-, P}^P(\lambda_R^+)$ by (2.3). Hence we have $T_{\lambda_R^+} = E_{o_-}(\lambda_R^+) = j_P^{-*}(E_{o_-, P}^P(\lambda_R^+)) = j_P^{-*}(T_{\lambda_R^+}^P)$ by Proposition 2.6. Therefore $e_G(\sigma)(T_{\lambda_R^+}) = \sigma(T_{\lambda_R^+}^P)$. Hence we have

$$A \simeq \{(x_n)_{n \in \mathbb{Z}_{\geq 0}} \mid x_{n+1} \sigma(T_{\lambda_R^+}^P) = x_n, x_n \in \sigma\}$$

Since $R \supset P_2$, we have $\Sigma^+ \setminus \Sigma_R^+ = \Sigma_P^+ \setminus \Sigma_{P \cap R}^+$. Therefore we can take λ_R^+ as $\lambda_{R \cap P}^{P+}$. Hence

$$\{(x_n)_{n \in \mathbb{Z}_{\geq 0}} \mid x_{n+1} \sigma(T_{\lambda_R^+}^P) = x_n\} \simeq B.$$

Namely there exists an isomorphism $A \simeq B$ as vector spaces which is characterized by $\varphi((T_{\lambda_R^+}^R)^{-n}) = \psi((T_{\lambda_R^+}^{P \cap R})^{-n})$ for any $n \in \mathbb{Z}_{\geq 0}$ where $\varphi \in A$ corresponds to $\psi \in B$ by this isomorphism.

We prove that this isomorphism is $(\mathcal{H}_{P \cap R}^{R-}, j_{P \cap R}^{R-*})$ -equivariant. Let $w \in W_{P \cap R}^{R-}(1)$. By the assumption $\Delta_{P_2} \subset \Delta_R$, $\Sigma_R^+ \setminus \Sigma_{P \cap R}^+ = \Sigma_{P_2}^+ = \Sigma^+ \setminus \Sigma_P^+$. Hence $w \in W_P^-(1)$. Take $k \in \mathbb{Z}_{\geq 0}$ such that $w(\lambda_R^+)^k$ is R -positive. Since φ is (\mathcal{H}_R^+, j_R^+) -equivariant, using Lemma 2.6, we have

$$\begin{aligned} (\varphi j_{P \cap R}^{R-*}(E_{o_-, P \cap R}^{P \cap R}(w)))((T_{\lambda_R^+}^R)^{-n}) &= \varphi(E_{o_-, R}^R(w)(T_{\lambda_R^+}^R)^{-n}) \\ &= \varphi(E_{o_-, R}^R(w(\lambda_R^+)^k)(T_{\lambda_R^+}^R)^{-(n+k)}) \\ &= \varphi((T_{\lambda_R^+}^R)^{-(n+k)} j_R^+(E_{o_-}(w(\lambda_R^+)^k))) \\ &= \varphi((T_{\lambda_R^+}^R)^{-(n+k)} e_G(\sigma)(E_{o_-}(w(\lambda_R^+)^k))). \end{aligned}$$

Since λ_R^+ is in the center of $W_R(1)$ and $P_2 \subset R$, λ_R^+ is also in the center of $W_{P_2}(1)$, hence we have $\langle \alpha, \nu(\lambda_R^+) \rangle = 0$ for any $\alpha \in \Delta_{P_2}$. Hence for any $\alpha \in \Sigma^+ \setminus \Sigma_P^+$, we have $\langle \alpha, \nu(\lambda_R^+) \rangle = 0$ since $\Sigma^+ \setminus \Sigma_P^+ = \Sigma_{P_2}^+$. Therefore λ_R^+ is both P -positive and P -negative. Recall that w is also P -negative. Therefore $w(\lambda_R^+)^k$ is also P -negative. Hence $e_G(\sigma)(E_{o_-}(w(\lambda_R^+)^k)) = \sigma(E_{o_-, P}^P(w(\lambda_R^+)^k))$ by the definition of the extension and Lemma 2.6. Therefore

$$\begin{aligned} \varphi((T_{\lambda_R^+}^R)^{-(n+k)} e_G(\sigma)(E_{o_-}(w(\lambda_R^+)^k))) &= \varphi((T_{\lambda_R^+}^R)^{-(n+k)} \sigma(E_{o_-, P}^P(w(\lambda_R^+)^k))) \\ &= \psi((T_{\lambda_R^+}^{P \cap R})^{-(n+k)} \sigma(E_{o_-, P}^P(w(\lambda_R^+)^k))). \end{aligned}$$

Since $w(\lambda_R^+)^k \in W_{P \cap R}(1)$ is R -positive, we have $w(\lambda_R^+)^k \in W_{P \cap R}^{P+}(1)$. Therefore $E_{o_-, P}^P(w(\lambda_R^+)^k) = j_{P \cap R}^{P+}(E_{o_-, P \cap R}^{P \cap R}(w(\lambda_R^+)^k))$ by Lemma 2.6. Hence

$$\begin{aligned} \psi((T_{\lambda_R^+}^{P \cap R})^{-(n+k)} \sigma(E_{o_-, P}^P(w(\lambda_R^+)^k))) &= \psi((T_{\lambda_R^+}^{P \cap R})^{-(n+k)} E_{o_-, P \cap R}^{P \cap R}(w(\lambda_R^+)^k)) \\ &= (\psi E_{o_-, P \cap R}^{P \cap R}(w))((T_{\lambda_R^+}^{P \cap R})^{-n}). \end{aligned}$$

Hence $A \simeq B$ as $(\mathcal{H}_{P \cap R}^{R-}, j_{P \cap R}^{R-*})$ -modules.

Let $w \in W_{P_2 \cap R, \text{aff}}(1)$. Then $w \in W_{P_2, \text{aff}}(1)$. Since $\Sigma^+ \setminus \Sigma_R^+ \subset \Sigma^+ \setminus \Sigma_{P_2}^+ = \Sigma_P^+$ and for any $\alpha \in \Sigma_P^+$ is orthogonal to elements in $\nu(\Lambda(1) \cap W_{P_2, \text{aff}}(1))$, w is both R -positive and R -negative. Hence by Corollary 2.7, we have $j_R^+(T_w^{R*}) = T_w^*$. Therefore, for $\varphi \in A$, we have

$$\begin{aligned} (\varphi T_w^{R*})((T_{\lambda_R^+}^R)^{-n}) &= \varphi(T_w^{R*}(T_{\lambda_R^+}^R)^{-n}) \\ &= \varphi((T_{\lambda_R^+}^R)^{-n} T_w^{R*}) \\ &= \varphi((T_{\lambda_R^+}^R)^{-n}) e_G(\sigma)(j_R^+(T_w^{R*})) \\ &= \varphi((T_{\lambda_R^+}^R)^{-n}) e_G(\sigma)(T_w^*) \\ &= \varphi((T_{\lambda_R^+}^R)^{-n}). \end{aligned}$$

Therefore $T_w^{R*} = 1$ on A . Hence $A \simeq e_R(B)$ by the definition of the extension. \square

Proof of Proposition 5.11. By Lemma 5.12, we have

$$\bigoplus_{Q_1 \supsetneq Q} R_R(I_{Q_1}(e_{Q_1}(\sigma))) \rightarrow R_R(I_Q(e_Q(\sigma))) \rightarrow R_R(\text{St}_Q(\sigma)) \rightarrow 0$$

By Proposition 5.1,

$$\bigoplus_{Q' \supsetneq Q} I_{Q' \cap R}^R(R_{Q' \cap R}^{Q'}(e_{Q'}(\sigma))) \rightarrow I_{Q \cap R}^R(R_{Q \cap R}^Q(e_Q(\sigma))) \rightarrow R_R(\text{St}_Q(\sigma)) \rightarrow 0.$$

If $\Delta_Q \neq \Delta_{Q \cap R} \cup \Delta_P$, then $R_{Q \cap R}^Q(e_Q(\sigma)) = 0$ by Lemma 5.17. Hence $R_R(\text{St}_Q(\sigma)) = 0$. Assume that $\Delta_Q = \Delta_{Q \cap R} \cup \Delta_P$. Then $R_{Q \cap R}^Q(e_Q(\sigma)) = e_{Q \cap R}(R_{P \cap R}^P(\sigma))$. Let $Q' \supsetneq Q$. If $\Delta_{Q'} = \Delta_{Q' \cap R} \cup \Delta_P$, then $R_{Q' \cap R}^{Q'}(e_{Q'}(\sigma)) = e_{Q' \cap R}(R_{P \cap R}^P(\sigma))$ and $Q' \cap R \supsetneq Q \cap R$. Otherwise, it is zero. Putting $Q'' = Q' \cap R$, we have

$$\begin{aligned} \bigoplus_{R \supset Q'' \supsetneq Q \cap R} I_{Q''}^R(e_{Q''}(R_{P \cap R}^P(\sigma))) &\rightarrow I_{Q \cap R}^R(e_{Q \cap R}(R_{P \cap R}^P(\sigma))) \\ &\rightarrow R_R(\text{St}_Q(\sigma)) \rightarrow 0. \end{aligned}$$

Therefore we have $R_R(\text{St}_Q(\sigma)) = \text{St}_{Q \cap R}^R(R_{P \cap R}^P(\sigma))$. \square

5.6. Supersingular modules. Assume that $p = 0$ in C .

Proposition 5.18. *Let $P \subsetneq G$ be a proper parabolic subgroup and π a supersingular \mathcal{H} -module. Then we have $L_P(\pi) = R_P(\pi) = 0$.*

We need a lemma.

Lemma 5.19. *Assume that $\lambda \in Z(\Lambda(1))$ satisfies that for any $w \in W_0$, we have $n_w \cdot \lambda = \lambda$ if and only if $w(\nu(\lambda)) = \nu(\lambda)$. Put $\mathcal{O} = W(1) \cdot \lambda$. Then for any orientation o and $n \in \mathbb{Z}_{\geq 0}$, we have $z_{\mathcal{O}}^n E_o(\lambda) = E_o(\lambda)^{n+1}$.*

Proof. Since λ is in the center of $\Lambda(1)$, we have $\mathcal{O} = \{n_w \cdot \lambda \mid w \in W_0\}$. We also have $\mathcal{O} = \{n_w \cdot \lambda \mid w \in W_0/\text{Stab}_{W_0}(\nu(\lambda))\}$ by the condition on λ . Therefore we have $z_{\mathcal{O}} = \sum_{w \in W_0/\text{Stab}_{W_0}(\nu(\lambda))} E_o(n_w \cdot \lambda)$. Hence

$$z_{\mathcal{O}}^2 = \sum_{v_1, v_2 \in W_0/\text{Stab}_{W_0}(\nu(\lambda))} E_o(n_{v_1} \cdot \lambda) E_o(n_{v_2} \cdot \lambda).$$

If $v_1, v_2 \in W_0$ does not belong to the same coset in $W_0/\text{Stab}_{W_0}(\nu(\lambda))$, then $v_1(\nu(\lambda))$ and $v_2(\nu(\lambda))$ are not in the same closed chamber. Hence $E_o(n_{v_1} \cdot \lambda) E_o(n_{v_2} \cdot \lambda) = 0$ by (2.1) and Lemma 2.11. Therefore

$$z_{\mathcal{O}}^2 = \sum_{v \in W_0/\text{Stab}_{W_0}(\nu(\lambda))} E_o(n_v \cdot \lambda)^2.$$

Inductively, we get

$$z_{\mathcal{O}}^n = \sum_{v \in W_0/\text{Stab}_{W_0}(\nu(\lambda))} E_o(n_v \cdot \lambda)^n.$$

Hence

$$z_{\mathcal{O}}^n E_o(\lambda) = \sum_{v \in W_0/\text{Stab}_{W_0}(\nu(\lambda))} E_o(n_v \cdot \lambda)^n E_o(\lambda).$$

If $v \notin \text{Stab}_{W_0}(\nu(\lambda))$, then $E_o(n_v \cdot \lambda) E_o(\lambda) = 0$. Hence

$$z_{\mathcal{O}}^n E_o(\lambda) = E_o(\lambda)^{n+1}.$$

We get the lemma. \square

Proof of Proposition 5.18. Let $\lambda = \lambda_{\bar{P}} \in Z(W_P(1))$ be as in Proposition 2.5. If $w \in W_0$ fixes $\nu(\lambda)$, then $w \in W_{0,P}$. Since $\lambda \in Z(W_P(1))$, we have $n_w \cdot \lambda = \lambda$. Namely, λ satisfies the condition of the above lemma. We also have $L_P(\pi) = \pi E_{o_-}(\lambda)^{-1}$.

Set $\mathcal{O} = W(1) \cdot \lambda$. Let $x \in \pi$. Since π is supersingular, there exists $n \in \mathbb{Z}_{>0}$ such that $x z_{\mathcal{O}}^n = 0$. Hence $x z_{\mathcal{O}}^n E_{o_-}(\lambda) = 0$. Therefore we have $x E_{o_-}(\lambda)^{n+1} = 0$ by the above lemma. Hence the image of x in $L_P(\pi) = \pi E_{o_-}(\lambda)^{-1}$ is zero. We get $L_P(\pi) = 0$.

Next we prove $R_P(\pi) = n_{w_G w_P}^{-1} \text{Hom}_{(\mathcal{H}_{P'}^+, j_{P'}^+)}(\mathcal{H}_{P'}, \pi) = 0$ where $P' = n_{w_G w_P} P^{\text{op}} n_{w_G w_P}^{-1}$. Let $\lambda = \lambda_{P'}^+ \in Z(W_{P'}(1))$ be as in Proposition 2.5. Then again λ satisfies the condition of the above lemma. Take $n \in \mathbb{Z}_{>0}$ such that $\pi(z_{\mathcal{O}}^n) = 0$. Let $\varphi \in \text{Hom}_{(\mathcal{H}_{P'}^+, j_{P'}^+)}(\mathcal{H}_{P'}, \pi)$ and $X \in \mathcal{H}_{P'}$. Since $j_{P'}^+(E_{o_-, P'}^{P'}(\lambda)) = E_{o_-}(\lambda)$ by Lemma 2.6, with the previous lemma, we have

$$\begin{aligned} \varphi(X) &= \varphi(X E_{o_-, P'}^{P'}(\lambda)^{-(n+1)}) j_{P'}^+(E_{o_-, P'}^{P'}(\lambda)^{n+1}) \\ &= \varphi(X E_{o_-, P'}^{P'}(\lambda)^{-(n+1)}) E_{o_-}(\lambda)^{n+1} \\ &= \varphi(X E_{o_-, P'}^{P'}(\lambda)^{-(n+1)}) z_{\mathcal{O}}^n E_{o_-}(\lambda) = 0. \end{aligned}$$

We get the proposition. \square

5.7. Simple modules. Assume that C is an algebraically closed field of characteristic p .

Theorem 5.20. *Let P be a parabolic subgroup, σ a simple supersingular right \mathcal{H}_P -module and Q a parabolic subgroup between P and $P(\sigma)$. For a parabolic subgroup R , we have*

$$L_R(I(P, \sigma, Q)) = \begin{cases} I_R(P, \sigma, Q \cap R) & (P \subset R, \Delta(\sigma) \subset \Delta_Q \cup \Delta_R), \\ 0 & (\text{otherwise}). \end{cases}$$

and

$$R_R(I(P, \sigma, Q)) = \begin{cases} I_R(P, \sigma, Q) & (Q \subset R), \\ 0 & (\text{otherwise}). \end{cases}$$

Proof. By Corollary 5.8, we have

$$L_R(I(P, \sigma, Q)) = L_R(I_{P(\sigma)}(\text{St}_Q^{P(\sigma)}(\sigma))) = I_{P(\sigma) \cap R}^R(L_{P(\sigma) \cap R}^{P(\sigma)}(\text{St}_Q^{P(\sigma)}(\sigma)))$$

If $\Delta_Q \cup (\Delta(\sigma) \cap \Delta_R) \neq \Delta(\sigma)$, then it is zero by Proposition 5.10. Since $Q \subset P(\sigma)$, we have $\Delta_Q \subset \Delta(\sigma)$. Hence $\Delta_Q = \Delta(\sigma) \cap \Delta_Q$ and, therefore we have $\Delta_Q \cup (\Delta(\sigma) \cap \Delta_R) = \Delta(\sigma) \cap (\Delta_Q \cup \Delta_R)$. Hence $\Delta_Q \cup (\Delta(\sigma) \cap \Delta_R) = \Delta(\sigma)$ if and only if $\Delta(\sigma) \subset \Delta_Q \cup \Delta_R$. Also from Proposition 5.10, if $\Delta(\sigma) \subset \Delta_Q \cup \Delta_R$, we have

$$L_R(I(P, \sigma, Q)) = I_{P(\sigma) \cap R}^R(\text{St}_{Q \cap R}^{P(\sigma) \cap R}(L_{P \cap R}^P(\sigma))),$$

here we use $Q \subset P(\sigma)$. By Proposition 5.18, this is zero if $P \cap R \neq P$, namely $P \not\subset R$. If $P \subset R$, then

$$L_R(I(P, \sigma, Q)) = I_{P(\sigma) \cap R}^R(\text{St}_{Q \cap R}^{P(\sigma) \cap R}(\sigma)) = I(P, \sigma, Q \cap R).$$

For the functor R , by Proposition 5.1, we have

$$R_R(I(P, \sigma, Q)) = R_R(I_{P(\sigma)}(\text{St}_Q^{P(\sigma)}(\sigma))) = I_{P(\sigma) \cap R}^R(R_{P(\sigma) \cap R}^{P(\sigma)}(\text{St}_Q^{P(\sigma)}(\sigma))).$$

By Proposition 5.11, this is zero if $\Delta_Q \neq \Delta_{Q \cap R \cap P(\sigma)} \cup \Delta_P$. Note that we have $Q \cap R \cap P(\sigma) = Q \cap R$ since $Q \subset P(\sigma)$. If $\Delta_Q = \Delta_{Q \cap R} \cup \Delta_P$, then we have

$$R_R(I(P, \sigma, Q)) = I_{P(\sigma) \cap R}^R(\text{St}_{Q \cap R}^{P(\sigma) \cap R}(R_{P \cap R}^P(\sigma))).$$

By Proposition 5.18, this is zero if $P \cap R \neq P$. If $P \cap R = P$, namely $P \subset R$, then we have

$$R_R(I(P, \sigma, Q)) = I_{P(\sigma) \cap R}^R(\text{St}_{Q \cap R}^{P(\sigma) \cap R}(\sigma)) = I_R(P, \sigma, Q \cap R).$$

Hence we get

$$R_R(I(P, \sigma, Q)) = \begin{cases} I_R(P, \sigma, Q \cap R) & (P \subset R, \Delta_Q = \Delta_{Q \cap R} \cup \Delta_P), \\ 0 & (\text{otherwise}). \end{cases}$$

If $P \subset R$, then $\Delta_{Q \cap R} \supset \Delta_P$. Hence $\Delta_Q = \Delta_{Q \cap R} \cup \Delta_P$ implies $\Delta_Q = \Delta_{Q \cap R}$, namely $Q \subset R$. On the other hand, if $Q \subset R$ then $P \subset R$ and $\Delta_Q = \Delta_Q \cup \Delta_P = \Delta_{Q \cap R} \cup \Delta_P$. We get the theorem. \square

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